

Relativistic point dynamics and Einstein formula as a property of localized solutions of a nonlinear Klein-Gordon equation.

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Abstract

Einstein's relation $E = Mc^2$ between the energy E and the mass M is the cornerstone of the relativity theory. This relation is often derived in a context of the relativistic theory for closed systems which do not accelerate. By contrast, Newtonian approach to the mass is based on an accelerated motion. We study here a particular neoclassical field model of a particle governed by a nonlinear Klein-Gordon (KG) field equation. We prove that if a solution to the nonlinear KG equation and its energy density concentrate at a trajectory, then this trajectory and the energy must satisfy the relativistic version of Newton's law with the mass satisfying Einstein's relation. Therefore the internal energy of a localized wave affects its acceleration in an external field as the inertial mass does in Newtonian mechanics. We demonstrate that the "concentration" assumptions hold for a wide class of rectilinear accelerating motions.

1 Introduction

One of the goals of this paper is to demonstrate that a certain field model of a particle provides for a deeper understanding of the Einstein energy-mass relation $E = Mc^2$. This field model is an integral part of our recently introduced neoclassical electromagnetic (EM) theory [2]-[5] of charges, and the relativistic aspects of that theory has been explored in [6]. An idea that a particle can be viewed as a field excitation carrying a certain amount of energy is a rather old one. Einstein and Infeld wrote in [18, p. 257]: "What impresses our senses as matter is really a great concentration of energy into a comparatively small space. We could regard matter as the regions in space where the field is extremely strong." But implementation of this idea in a mathematically sound theory is a challenging problem. Einstein remarks in his letter to Ernst Cassirer in March 16, 1937, [37, pp. 393-394]: "One must always bear in mind that up to now we know absolutely nothing about the laws of motion of material points from the standpoint of "classical field theory." For the mastery of this problem, however, no special physical hypothesis is needed, but "only" the solution of certain mathematical problems".

We start with recalling basic facts from the relativistic mechanics including the relativistic dynamics of a mass point and the relativistic field theory, see for instance, [1], [8], [27], [30],

[36]. In a relativistic field theory the relativistic field dynamics is derived from a relativistic covariant Lagrangian. The field equations, the energy and the momentum, the forces and their densities are naturally defined in terms of the Lagrangian both in the cases of closed and non closed (with external forces) systems. For a closed system the total energy-momentum (E, \mathbf{P}) transforms as 4-vector, [1, Sec. 7.1-7.5], [27, Sec. 3.1-3.3, 3.5], [30, Sec. 37], [36, Sec. 4.1]. In particular, the total momentum \mathbf{P} has a simple form $\mathbf{P} = M\mathbf{v}$ where the constant velocity \mathbf{v} originates from the corresponding parameter of the Lorentz group. Then one can naturally define and interpret the mass M for a closed relativistic system as the coefficient of proportionality between the momentum \mathbf{P} and the velocity \mathbf{v} . As to the energy of a closed system, the celebrated Einstein energy-mass relation holds

$$E = Mc^2, \quad M = m_0\gamma, \quad \gamma = (1 - \mathbf{v}^2/c^2)^{-1/2}, \quad (1)$$

where m_0 is the rest mass and γ is the Lorentz factor. Observe that the above definition of mass is based on the relativistic argument for a uniform motion with a constant velocity \mathbf{v} without acceleration. An immediate implication of Einstein's mass-energy relation (1) is that the rest mass of a closed system is essentially equivalent to the internal energy of the system.

Let us turn now to the relativistic dynamics of a mass point which accelerates under the action of a force $\mathbf{f}(t, \mathbf{r})$. This dynamics is governed by the relativistic version of Newton's equation of the form, [8], [30], [27]:

$$\frac{d}{dt}(M\mathbf{v}(t)) = \mathbf{f}(t, \mathbf{r}), \quad \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}, \quad M = m_0\gamma. \quad (2)$$

Note that in (2) the rest mass m_0 is prescribed as an intrinsic property of the mass point, for the mass point does not have internal degrees of freedom. The equation (2), just as in the case of classical Newtonian mechanics, suggests that the mass M is a measure of inertia, that is the coefficient that relates the acceleration to the known force \mathbf{f} . To summarize, according to the relativity principles the rest mass is naturally defined for a uniform motion, whereas in the Newtonian mechanics the concept of inertial mass is introduced through an accelerated motion. Note that the relativistic and non-relativistic masses are sometimes considered to be "rival and contradictory", [19].

A principal problem we want to take on here is as follows. We would like to construct a field model of a charge where the internal energy of a localized wave affects its acceleration in an external field the same way the inertial mass does in Newtonian mechanics. We could not find such a model in literature and we introduce and study it here. The model allows, in particular, to consider the uniform motion in the absence of external forces and the accelerated motion in the same framework. Hence within the same framework (but in different regimes) we can determine the mass either from the analysis of the 4-vector of the energy-momentum or using the Newtonian approach. When both the approaches are relevant, they agree, see Remark 3.7.

The proposed model is based on our manifestly relativistic Lagrangian field theory [2]-[5]. In the case of a single charge its state is described by a complex-valued scalar field (wave function) $\psi(t, \mathbf{x})$ of the time t and the position vector $\mathbf{x} \in \mathbb{R}^3$ and its time evolution is governed by the following nonlinear Klein-Gordon (KG) equation

$$-\frac{1}{c^2}\tilde{\partial}_t^2\psi + \tilde{\nabla}^2\psi - G'(|\psi|^2)\psi - \frac{m^2c^2}{\chi^2}\psi = 0, \quad (3)$$

where m is a positive mass parameter and χ is a constant which in physical applications coincides with (or is close to) the Planck constant \hbar . The covariant derivatives in equation (3) are defined by

$$\tilde{\partial}_t = \partial_t + \frac{iq}{\chi}\varphi, \quad \tilde{\nabla} = \nabla - \frac{iq}{\chi c}\mathbf{A}, \quad (4)$$

where q is the value of the charge and $\varphi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$ are the potentials of the *external EM field*. The external EM field acts upon the charge resulting ultimately in the Lorentz force when Newton's point mass equation is relevant. In this paper, for simplicity, we treat the case where only electric forces are present and the magnetic potential is set to be zero, that is

$$\mathbf{A} = \mathbf{0}, \quad \tilde{\nabla} = \nabla. \quad (5)$$

The physical treatment of the case with a non-zero external magnetic field is provided in [6].

The nonlinear term G' in the KG (3) provides for the existence of localized solutions for resting or uniformly moving charges. The nonlinearity G and its properties are considered in Section 2.1. Importantly, the nonlinearity G involves a size parameter a that determines the spatial scale of the charge when at rest. Notice that $|\psi|^2$ is interpreted as a charge distribution and not as a probability density.

A sketch of our line of argument is as follows. From the Lagrangian of the KG equation introduced below we derive a usual expression for the energy density which allows to define the energy involved in Einstein's formula. To be able to apply the Newtonian approach it is necessary to relate point trajectories $\mathbf{r}(t)$ to wave solutions ψ and trace point dynamics to the field equations. The latter is accomplished based on a concept of solutions "concentrating at a trajectory".

Roughly speaking (see Section 3.1 for the exact definitions) solutions concentrate at a given trajectory $\hat{\mathbf{r}}(t)$ if their energy densities $\mathcal{E}(\mathbf{x}, t)$ restricted to R_n -neighborhoods of $\hat{\mathbf{r}}(t)$ locally converge to $E(t)\delta(\mathbf{x} - \hat{\mathbf{r}}(t))$ as $R_n \rightarrow 0$. A concise formulation of the main result of the paper, Theorem 3.6, is as follows. We prove that *if a sequence of solutions of the KG equation concentrates at a trajectory $\hat{\mathbf{r}}(t)$ then the restricted energy $\bar{\mathcal{E}}_n$ of the solutions converges to a function $\bar{\mathcal{E}}(t)$ so that the following relativistic version of Newton's equation holds:*

$$\frac{d}{dt} \left(\frac{\bar{\mathcal{E}}}{c^2} \hat{\mathbf{v}} \right) = \mathbf{f}(t, \hat{\mathbf{r}}), \quad \hat{\mathbf{v}} = \frac{d}{dt} \hat{\mathbf{r}}. \quad (6)$$

The electric force \mathbf{f} in (6) is defined by formula

$$\mathbf{f}(t, \hat{\mathbf{r}}) = -\bar{\rho} \nabla \varphi(t, \hat{\mathbf{r}}) \quad (7)$$

where φ is the electric potential as in the KG equation (4), $\bar{\rho}$ is a constant describing the limit charge. The limit restricted energy $E(t)$ satisfies relation

$$\frac{\bar{\mathcal{E}}(t)}{c^2} = M_0 \gamma, \quad \gamma = (1 - \hat{\mathbf{v}}^2/c^2)^{-1/2}, \quad (8)$$

where M_0 is a constant which plays the role of a generalized rest mass. Observe that *equation (6) takes the form of the relativistic version of Newton's law if the mass is defined by Einstein's formula $M(t) = \frac{1}{c^2} \bar{\mathcal{E}}(t)$* . The relation between the generalized rest mass M_0 and the rest mass m_0 of resting solutions is discussed in Remark 3.7. Note that the same KG equation (3) with the same value of the mass parameter m has rest solutions with different energies, and consequently different rest masses, see Sections (2.3) and 2.4.

Therefore we can make the following conclusion: *in the framework of the KG field theory, the relativistic material point dynamics is represented by concentrating solutions of the KG equation with the mass determined by Einstein's formula from the limit restricted energy of the solutions.*

We can add to the above outline a few guiding points to the mathematical aspects of our approach. Equation (6) in Theorem 3.6 produces a *necessary condition* for solutions of KG equation to concentrate to a trajectory and reveals their asymptotic point-like dynamics. To obtain this necessary condition we have to derive the point dynamics governed by an ordinary differential equation (6) from the dynamics of waves governed by a partial differential equation. The concept of "concentration of functions at a given trajectory $\hat{\mathbf{r}}(t)$ ", see Definition 3.3 below, is the first step in relating spatially localized fields ψ to point trajectories. The definition of concentration of functions has a sufficient flexibility to allow for general regular trajectories $\hat{\mathbf{r}}(t)$ and plenty of functions localized about the trajectory, see Example 3.4. But if a sequence of functions concentrating at a given trajectory *are also solutions of the KG equation then, according to Theorem 3.6, the trajectory and the limit energy must satisfy the relativistic Newton's equations together with Einstein's formula.* To derive the equations (6) and (8), we introduce the energy $\bar{\mathcal{E}}_n(t)$ restricted to a narrow tubular neighborhood of the trajectory with radius R_n and adjacent ergocenters $\mathbf{r}_n(t)$ of the concentrating solutions; then we infer integral equations for the restricted energy and adjacent ergocenters from the energy and momentum conservation laws and the continuity equation, and then we pass to the limit as $R_n \rightarrow 0$, see Theorems 3.11 and 3.15 and the proof of Theorem 3.6. The intrinsic length scales a and $a_C = \frac{\hbar}{mc}$ of concentrating solutions are much smaller than the radius R_n , namely $R_n/a \rightarrow \infty$, $R_n/a_C \rightarrow \infty$. The determination of restricted energy and charge and adjacent ergocenters involves integration over a large relative to a and a_C spatial domain of radius R_n . Therefore, equation (6) inherits integral, non-local characteristics of concentrating solutions which cannot be reduced to their behavior on the trajectory, namely the limit restricted energy $\bar{\mathcal{E}}(t)$ and restricted charge $\bar{\rho}_\infty$. Therefore it is natural to call our method of determination of the point trajectories *semi-local*. This semi-local feature allows to capture Einstein's relation between mass and energy.

Note that in many problems of physics and mathematics a common way to establish a relation between point dynamics and wave dynamics is by means of the WKB method, see for instance [26], [29, Sec. 7.1]. We remind that the WKB method is based on the quasiclassical ansatz for solutions to a hyperbolic partial differential equation and their asymptotic expansion. The leading term of the expansion results in the eikonal equation; wavepackets and their energy propagate along its characteristics. The characteristics represent point dynamics and are determined from a system of ODE which can be interpreted as a law of motion or a law of propagation. The construction of the characteristics involves only local data. The proposed here approach also relates waves governed by certain PDE's in asymptotic regimes to the point dynamics but it differs significantly from the WKB method. In particular, our approach is not based on any specific ansatz and it is not not entirely local but rather it is semi-local.

A simple and important example of concentrating solutions is provided by a resting or uniformly moving charge in Example 3.5. Examples of *accelerating* charges are constructed in Section 4, where we show that all the assumptions on concentrating solutions imposed in the Theorem 3.6 are satisfied. For a given accelerating rectilinear translational motion and a fixed Gaussian shape of $|\psi|$ we construct an electric potential φ consisting of (i) an explicitly written principal component yielding desired acceleration and (ii) an additional

vanishingly small "balancing" component allowing for the shape $|\psi|$ to be exactly preserved. The construction of the balancing component is reduced to solving a system of characteristic ODE which allows for a detailed analysis.

The rest of the paper is structured as follows. In Section 2.1 we describe the nonlinearities G . In Section 2.2 we introduce the Lagrangian for the KG equation and related densities of charge, energy and momentum, and verify then the corresponding conservation laws. In Sections 2.3 and 2.4 solutions for resting and uniformly moving charge are considered and their energies are found. In Section 3 we define our key concepts and derive then the equations (6), (8) as necessary conditions for solutions of the KG equation to concentrate at a trajectory. In Section 4 an example is provided for a relativistically accelerating charge which satisfies all the assumptions imposed in the theorem on concentrating solutions. In Section 5 we discuss application of the results of the paper to linear KG equations. Note that the generalization of the results of the paper to higher than 3 space dimensions is straightforward.

2 Basic properties of the Klein-Gordon equation

2.1 Nonlinearity, its basic properties and examples

In this section we describe the nonlinearity $G'(|\psi|^2)$ which enters KG equation (3). We assume that $G'(s)$ is of class $C^1(\mathbb{R} \setminus 0)$, it may have a mild singularity point at $s = 0$. Namely, we assume that the function $G'(|\psi|^2)\psi$, $\psi \in \mathbb{C}$, can be extended to a function which belongs to the Holder class $C^\alpha(\mathbb{C})$ with any α such that $1 > \alpha > 0$ and the antiderivative $G(|\psi|^2)$ belongs to $C^{1+\alpha}(\mathbb{C})$. Below we give several explicit examples of the function $G'(|\psi|^2)$ which allows for resting, time-harmonic localized solutions of (3) and satisfy the above assumptions. In the examples we define the nonlinearity G and its dependence on the size parameter a based on the ground state $\dot{\psi} \geq 0$. The dependence of the ground state $\dot{\psi}$ on the size parameter $a > 0$ is as follows:

$$\dot{\psi}(r) = \dot{\psi}_a(r) = a^{-3/2} \dot{\psi}_1(a^{-1}r), r = |x| \geq 0 \quad (9)$$

where $\dot{\psi}_1(r)$ is a given function. The dependence on a is chosen so that L^2 -norm $\|\dot{\psi}_a(|x|)\|$ does not depend on a , hence the function $\dot{\psi}_a(r)$ satisfies the normalization condition

$$\|\dot{\psi}_a(|x|)\| = \nu, \nu > 0$$

with a fixed ν for every $a > 0$. The function $\dot{\psi}_a(r)$ is assumed to be a smooth (three times continuously differentiable) positive monotonically decreasing function of $r \geq 0$ which is square integrable with weight r^2 , we assume that its derivative $\dot{\psi}'_a(r)$ is negative for $r > 0$ and we assume it to satisfy a charge normalization condition.

We assume that $\dot{\psi}_a$ satisfies the charge equilibrium equation:

$$\nabla^2 \dot{\psi}_a = G'_a(\dot{\psi}_a^2) \dot{\psi}_a. \quad (10)$$

This equation is obtained from (3) by substitution $\psi = e^{-i\omega_0 t} \dot{\psi}(|x|)$, $\omega_0 = mc^2/\chi$; this equation can be used to define the nonlinearity. Obviously, it is sufficient to define G'_a for $a = 1$ and then set $G'_a(s) = a^{-2} G'_1(a^3 s)$. From the equation (10) we express $G'_1(s)$ with $s = \dot{\psi}_1^2(r)$; since $\dot{\psi}_1^2(r)$ is a *monotonic* function, we can find its inverse $r = r(s)$, and we obtain an explicit expression

$$G'_1(s) = \nabla^2 \dot{\psi}_1(r(s)) / \dot{\psi}_1(r(s)), \quad 0 = \dot{\psi}_1^2(\infty) \leq s \leq \dot{\psi}_1^2(0). \quad (11)$$

Since $\mathring{\psi}_1(r)$ is smooth and $\partial_r \mathring{\psi}_1 < 0$, $G'(|\psi|^2)$ is smooth for $0 < |\psi|^2 < \mathring{\psi}_1^2(0)$. We assume that $\mathring{\psi}_a(r)$ is a smooth function of class C^3 , and we assume that we define an extension of $G'(s)$ for $s \geq \mathring{\psi}_1^2(0)$ as a function of class C^1 for all $r > 0$. Though $G'(s)$ may have a singular point at $s = 0$, we assume that the function $G'(|\psi|^2)\psi$ can be extended to a function of Holder class $C^\alpha(\mathbb{C})$ for any α , $0 < \alpha < 1$ and the antiderivative $G_1(|\psi|^2)$,

$$G_1(s) = \int_0^s G'_1(s') ds', \quad G_a(s) = a^{-5} G_1(a^3 s),$$

is a function of class $C^{1+\alpha}(\mathbb{C})$ for any $\alpha < 1$ with respect to the variable $\psi \in \mathbb{C}$. In particular, we assume that for every C_1 there exists such constant C that the following inequalities hold:

$$|G_1(|\psi|^2)| \leq C |\psi|^{1+\alpha}, \quad |G'_1(|\psi|^2)| |\psi| \leq C |\psi|^\alpha \quad \text{for } |\psi| \leq C_1. \quad (12)$$

Example 2.1 Let the form factor $\mathring{\psi}_1(r)$ decay as a power law, namely $\mathring{\psi}_1(r) = c_{\text{pw}}(1+r^2)^{-p}$, $p > 3/4$. Then

$$G'_1(\mathring{\psi}_1^2) = -2p \left(\mathring{\psi}_1 / c_{\text{pw}} \right)^{2/p} \left(\left(\mathring{\psi}_1 / c_{\text{pw}} \right)^{1/p} + (2p+2) \left(\mathring{\psi}_1 / c_{\text{pw}} \right)^{2/p} \right) \quad (13)$$

where c_{pw} is the normalization factor. The same formula (13) can be used to obtain extension $G'(s)$ for all $s \geq 0$. In this example $G'_1(s)$ is differentiable for all $s \geq 0$ if $3/p \geq 2$.

Example 2.2 Consider an exponentially decaying form factor $\mathring{\psi}_1$ of the form

$$\mathring{\psi}_1(r) = c_e e^{-(r^2+1)^p}. \quad (14)$$

A direct computation yields

$$\begin{aligned} G'(\mathring{\psi}^2) &= -2p \left((2p+1) \ln^{(p-1)/p} \left(c_e / \mathring{\psi} \right) - 2p \ln^{(2p-1)/p} \left(c_e / \mathring{\psi} \right) \right) \\ &\quad - 2p \left(2p \ln^{(2p-2)/p} \left(c_e / \mathring{\psi} \right) + (2-2p) \ln^{(p-2)/p} \left(c_e / \mathring{\psi} \right) \right). \end{aligned} \quad (15)$$

In particular, for $p = 1/2$ we obtain an exponentially decaying ground state $\mathring{\psi}_1(r) = c_e e^{-(r^2+1)^{1/2}}$ and for $s \leq c_e^2 e^{-2}$

$$G'_1(s) = 1 - \frac{4}{\ln(c_e^2/s)} - \frac{4}{\ln^2(c_e^2/s)} - \frac{8}{\ln^3(c_e^2/s)}. \quad (16)$$

We can extend it for larger s as follows:

$$G'_1(s) = G'_1(c_e^2 e^{-2}) = -3 \quad \text{if } s \geq 2c_e^2 e^{-2}. \quad (17)$$

and in the interval $c_e^2 e^{-2} \leq s \leq 2c_e^2 e^{-2}$ interpolate to obtain a smooth function for $s > 0$. The function $G'_1(s)$ defined by (16) has a limit at $s = 0$ and can be extended to a continuous function, but is not differentiable at $s = 0$.

Example 2.3 *The Gaussian ground state is given by the formula*

$$\mathring{\psi}(r) = C_g e^{-r^2/2}, \quad C_g = \pi^{-3/4}. \quad (18)$$

Such a ground state is called gausson in [12]. The expression for G' can be derived from (15) in the particular case $p = 1$, or can be evaluated directly:

$$\nabla^2 \mathring{\psi}(r) / \mathring{\psi}(r) = r^2 - 3 = -\ln \left(\mathring{\psi}^2(r) / C_g^2 \right) - 3.$$

Hence, we define the nonlinearity by the formula

$$G'(|\psi|^2) = -\ln(|\psi|^2 / C_g^2) - 3, \quad (19)$$

and refer to it as the logarithmic nonlinearity. Dependence on the size parameter $a > 0$ is given by the formula

$$G'_a(|\psi|^2) = -a^{-2} \ln(a^3 |\psi|^2 / C_g^2) - 3a^{-2} \quad (20)$$

with the antiderivative

$$G(s) = G_a(s) = -a^{-2} s [\ln(a^3 s) + \ln \pi^{3/2} + 2], \quad s \geq 0. \quad (21)$$

2.2 Conservation laws for Klein-Gordon equation

The Lagrangian for the KG equation (3) is the following relativistic and gauge invariant expression

$$\mathcal{L}_1(\psi) = \frac{\chi^2}{2m} \left\{ \frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* - \tilde{\nabla} \psi \cdot \tilde{\nabla} \psi^* - \kappa_0^2 \psi^* \psi - G(\psi^* \psi) \right\} \quad (22)$$

where $\psi(t, \mathbf{x})$ is a complex valued wave function, and ψ^* is its complex conjugate. In the expression (22) c is the speed of light,

$$\kappa_0 = mc/\chi, \quad (23)$$

and the covariant derivatives in (22) are defined by (4). The nonlinear KG equation (3) is the Euler-Lagrange field equation for the Lagrangian (22).

Using the Noether theorem and the invariance of the Lagrangian (22) with respect to Lorentz and gauge transformations one can derive expressions for a number of conserved quantities, [2]-[5]. In particular, the charge and the current densities (if $\mathbf{A} = \mathbf{0}$) are as follows:

$$\rho = -\frac{\chi q}{mc^2} |\psi|^2 \operatorname{Im} \frac{\tilde{\partial}_t \psi}{\psi} = -\left(\frac{\chi q}{mc^2} \operatorname{Im} \frac{\partial_t \psi}{\psi} + \frac{q^2}{mc^2} \varphi \right) |\psi|^2, \quad (24)$$

$$\mathbf{J} = \frac{\chi q}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) = \frac{\chi q}{m} |\psi|^2 \operatorname{Im} \frac{\nabla \psi}{\psi}, \quad (25)$$

and the above expressions are exactly the charge and the current sources in the Maxwell equations, [2]-[5]. The expression for the energy, momentum and external force densities are respectively

$$\mathcal{E}(\psi) = \frac{\chi^2}{2m} \left[\frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* + \nabla \psi \cdot \nabla \psi^* + G(\psi^* \psi) + \kappa_0^2 \psi \psi^* \right], \quad (26)$$

$$\mathbf{P} = -\frac{\chi^2}{2mc^2} \left(\tilde{\partial}_t \psi \tilde{\nabla}^* \psi^* + \tilde{\partial}_t^* \psi^* \tilde{\nabla} \psi \right), \quad (27)$$

$$\mathbf{F} = \frac{\partial}{\partial x} \mathcal{L}_1 = \frac{\chi q}{2mc^2} i \left(-\psi^* \tilde{\partial}_t \psi + \tilde{\partial}_t^* \psi^* \psi \right) \nabla \varphi = -\rho \nabla \varphi. \quad (28)$$

Consequently, the total charge $\bar{\rho}$ and the energy $\bar{\mathcal{E}}$ are

$$\bar{\rho} = \int_{\mathbb{R}^3} \rho(t, \mathbf{x}) \, d^3x, \quad \bar{\mathcal{E}} = \frac{\chi^2}{2m} \int_{\mathbb{R}^3} \mathcal{E} \, d^3x. \quad (29)$$

The charge conservation/continuity equation

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0 \quad (30)$$

is readily verified by multiplying the both sides of equation (3) by ψ^* and taking the imaginary part. The equation (30) in turn implies the total charge conservation:

$$\int_{\mathbb{R}^3} \rho(t, \mathbf{x}) \, d^3x = \bar{\rho} = \text{const} = q. \quad (31)$$

The value $\bar{\rho} = q$ is taken to ensure that Coulomb's potential with this density has asymptotics $q/|\mathbf{x}|$ as $|\mathbf{x}| \rightarrow \infty$, [2]–[4].

The energy conservation equation

$$\partial_t \mathcal{E} = -c^2 \nabla \cdot \mathbf{P} - \nabla \varphi \cdot \mathbf{J} \quad (32)$$

can be verified by multiplying (3) by $\tilde{\partial}_t^* \psi^*$ defined by (4), taking the real part and carrying out elementary transformations, see Appendix 6. Similarly, the momentum conservation law takes the form

$$\partial_t \mathbf{P} - \mathbf{F} + \nabla \mathcal{L}_1(\psi) + \frac{\chi^2}{2m} \sum_j \partial_j (\partial_j \psi \nabla \psi^* + \partial_j \psi^* \nabla \psi) = 0, \quad (33)$$

where $\mathcal{L}_1(\psi)$ and \mathbf{F} are defined respectively by (22) and (28). It can be verified by multiplying (3) by $\tilde{\nabla}^* \psi^*$, taking the real part and carrying out elementary transformations, see Appendix 6. Integrating (32) and (33) with respect to \mathbf{x} we obtain equations for the total energy and momentum, namely

$$\partial_t \bar{\mathcal{E}} = - \int_{\mathbb{R}^3} \nabla \varphi \cdot \mathbf{J} \, d^3x, \quad \partial_t \int_{\mathbb{R}^3} \mathbf{P} \, d^3x = \int_{\mathbb{R}^3} \mathbf{F} \, d^3x. \quad (34)$$

Obviously, the total energy and momentum are preserved if the external force field $\nabla \varphi = \mathbf{0}$.

2.3 Rest states and their energies

When the external EM field vanishes, that is $\varphi = 0$, $\mathbf{A} = 0$, we define the rest states ψ as time harmonic solutions to the KG equation (3)

$$\psi(t, \mathbf{x}) = e^{-i\omega t} \check{\psi}(\mathbf{x}) \quad (35)$$

where $\check{\psi}(\mathbf{x})$ is assumed to be central-symmetric. The substitution of (35) in the KG equation (3) yields the following *nonlinear eigenvalue problem*

$$\nabla^2 \check{\psi} = G'_a(|\check{\psi}|^2) \check{\psi} + c^{-2} (\omega_0^2 - \omega^2) \check{\psi} = 0. \quad (36)$$

The solution $\check{\psi}$ must also satisfy the charge normalization condition (31) which takes the form

$$\int |\check{\psi}|^2 d^3x = \frac{\omega_0}{\omega}, \quad \omega_0 = \frac{mc^2}{\chi}. \quad (37)$$

The energy defined by (26), (29) yields for the standing wave (35) the following expression

$$\bar{\mathcal{E}} = \frac{\chi^2}{2m} \int_{\mathbb{R}^3} \left[\frac{1}{c^2} \omega^2 \check{\psi} \check{\psi}^* + \kappa_0^2 \check{\psi} \check{\psi}^* + \nabla \check{\psi} \nabla \check{\psi}^* + G_a(\check{\psi} \check{\psi}^*) \right] d^3x. \quad (38)$$

The problem (36), (37) has a sequence of solutions with the corresponding sequence of frequencies ω . Their energies $\bar{\mathcal{E}}_{0\omega}$ are related to the frequency ω by the formula

$$\bar{\mathcal{E}}_{0\omega} = \chi \omega (1 + \Theta(\omega)), \quad (39)$$

$$\Theta(\omega) = \Theta_0 \frac{a_C^2 \omega_0^2}{a^2 \omega^2}, \quad a_C = \frac{\chi}{mc}. \quad (40)$$

Here the coefficient $\Theta_0 = \frac{1}{3} \left\| \nabla \psi_1 \right\|^2$ depends on the shape of the rest charge, the parameter $a_C = \frac{\hbar}{mc} = \frac{\lambda_C}{2\pi}$ coincides in the physical applications with the *reduced Compton wavelength* of a particle with a mass m if $\chi = \hbar$. For the case of the logarithmic nonlinearity the ground state Gaussian shape is given by (18) and $\Theta_0 = 1/2$.

In the case of the logarithmic nonlinearity the original nonlinear eigenvalue problem (36) can be reduced by a change of variables to the following nonlinear eigenvalue problem with only one eigenvalue parameter ξ and the parameter-independent constraint:

$$\nabla^2 \check{\psi}_1 = G'_1(|\check{\psi}_1|^2) \check{\psi}_1 - \xi \check{\psi}_1, \quad \int_{\mathbb{R}^3} |\check{\psi}_1|^2 d^3x = 1. \quad (41)$$

The parameter ξ is related to the parameters in (36) by the formula

$$\xi = \frac{a^2}{a_C^2} \left(\frac{\omega^2}{\omega_0^2} - 1 \right) - \frac{1}{2} \ln \frac{\omega^2}{\omega_0^2}. \quad (42)$$

The eigenvalue problem (41) has infinitely many solutions (ξ_n, ψ_{1n}) , $n = 0, 1, 2, \dots$, representing localized charge distributions. The energy of ψ_n , $n > 0$, is higher than the energy of the Gaussian ground state which corresponds to $\xi = \xi_0 = 0$ and has the lowest possible energy. These solutions coincide with critical points of the energy functional under the constraint, for mathematical details see [14], [10], [11]. The next two values of ξ for the radial rest states are approximately 2.17 and 3.41 according to [13].

Applying to the rest solution the Lorentz transformation, one can easily obtain a solution which represents the charge moving with a constant velocity \mathbf{v} (see [2], [3] and the following Section 2.4).

2.4 Uniform motion of a charge

The free motion of a charge is governed by the KG equation (3) with vanishing external EM field, that is $\varphi = 0$, $\mathbf{A} = 0$. Since the KG equation is relativistic covariant, the solution can be obtained from a rest solution defined by (35), (36) by applying the Lorentz boost

transformation as in [2], [4]. Consequently, the solution to the KG equation (3) for a free charged particle that moves with a constant velocity \mathbf{v} takes the form

$$\psi(t, \mathbf{x}) = \psi_{\text{free}}(t, \mathbf{x}) = e^{-i(\gamma\omega t - \mathbf{k} \cdot \mathbf{x})} \check{\psi}(\mathbf{x}'), \quad (43)$$

with $\check{\psi}(\mathbf{x}')$ satisfying equation (36) and

$$\check{\psi}(\mathbf{x}') = \check{\psi}_a(\mathbf{x}') = a^{-3/2} \check{\psi}_1(\mathbf{x}'/a), \quad (44)$$

$$\mathbf{x}' = \mathbf{x} + \frac{(\gamma - 1)}{v^2} (\mathbf{v} \cdot \mathbf{x}) \mathbf{v} - \gamma \mathbf{v} t, \quad \mathbf{k} = \gamma \omega \frac{\mathbf{v}}{c^2}, \quad (45)$$

where γ is the Lorentz factor

$$\gamma = (1 - \beta^2)^{-1/2}, \quad \beta = c^{-1} \mathbf{v}. \quad (46)$$

Consequently, all quantities of interest for a free charge can be written explicitly. Namely, the charge density ρ defined by the relation (24), the total charge $\bar{\rho}$ and the total energy $\bar{\mathcal{E}}$ equal respectively

$$\rho = \gamma q |\check{\psi}(\mathbf{x}')|^2, \quad \bar{\rho} = \int \rho(\mathbf{x}) d^3x = q, \quad \bar{\mathcal{E}} = \gamma m c^2 (1 + \Theta(\omega)), \quad (47)$$

where $\Theta(\omega)$ is given by (40). The current density and the total current $\bar{\mathbf{J}}$ for the free charge equal respectively

$$\mathbf{J} = \frac{q}{m} \chi \operatorname{Im} \frac{\nabla \psi}{\psi} |\psi|^2 = \gamma \chi \frac{q}{m} \omega \frac{1}{c^2} \mathbf{v} |\check{\psi}|^2(\mathbf{x}'), \quad (48)$$

$$\bar{\mathbf{J}} = \int_{\mathbb{R}^3} \mathbf{J}(\mathbf{x}) d^3x = q \mathbf{v}. \quad (49)$$

The total momentum is given by the following formula

$$\bar{\mathbf{P}} = \gamma \mathbf{v} m (1 + \Theta(\omega)) = M \mathbf{v}, \quad M = \gamma m (1 + \Theta(\omega)). \quad (50)$$

Given the above kinematic representation $\bar{\mathbf{P}} = M \mathbf{v}$ of the momentum, it is natural, [27, Sec. 3.3], [30, Sec. 37], [9], to identify the coefficient M as the mass and to define the *rest mass* m_0 of the charge by the formula

$$m_0 = m (1 + \Theta(\omega)), \quad \Theta(\omega) = \Theta_0 a_C^2 / a^2, \quad (51)$$

and the expression (47) for the energy takes the form of Einstein's mass-energy relation $\bar{\mathcal{E}} = M c^2$.

A direct comparison shows that the above definition based on the Lorentz invariance of a uniformly moving free charge is fully consistent with the definition of the inertial mass which is derived from the analysis of the accelerated motion of localized charges in an external EM field in the following section, see Remark 3.7 for a more detailed discussion.

3 Relativistic dynamics of localized solutions

The point dynamics described by (2) involves only fields exactly at the location of the point. Therefore it is natural to make assumptions of localization of KG equations and their solutions which should involve only behavior of the fields in a vicinity of the trajectory. The speed of light c , the charge q and the mass parameter m are assumed to be fixed. There are two intrinsic length scales relevant for the nonlinear KG equation: the size parameter a that enters the nonlinearity and the quantity $a_C = \frac{\chi}{mc}$ known as the reduced Compton wavelength (if χ equals the Planck constant \hbar). In our analysis we suppose the parameter a_C and the size parameter a to become vanishingly small by taking a sequence of values $\chi_n \rightarrow 0$, $a_n \rightarrow 0$, $n = 1, 2, \dots$, when the corresponding values of the potential φ_n and solutions $\psi_n(t, \mathbf{x})$ of the KG equation are defined in contracting neighborhoods of a trajectory $\hat{\mathbf{r}}(t)$.

3.1 Concentration at a trajectory

In this section we define and give examples of *solutions concentrating at a trajectory*. A trajectory $\hat{\mathbf{r}}(t)$, $T_- \leq t \leq T_+$ is a twice continuously differentiable function with values in \mathbb{R}^3 satisfying

$$|\hat{\mathbf{v}}|, |\partial_t \hat{\mathbf{v}}| \leq C, \text{ where } \hat{\mathbf{v}}(t) = \partial_t \hat{\mathbf{r}}(t), \quad T_- \leq t \leq T_+. \quad (52)$$

In this section the trajectory $\hat{\mathbf{r}}(t)$ is assumed to be fixed. Being given a trajectory $\hat{\mathbf{r}}(t)$, we consider an associated with it family of neighborhoods contracting to it. Namely, we introduce the ball of radius R centered at $r = \hat{\mathbf{r}}(t)$:

$$\Omega(\hat{\mathbf{r}}(t), R) = \{\mathbf{x} : |\mathbf{x} - \hat{\mathbf{r}}(t)|^2 \leq R^2\} \subset \mathbb{R}^3, \quad R > 0 \quad (53)$$

and then for a sequence of positive $R_n \rightarrow 0$ we consider the sequence

$$\Omega_n = \Omega_n(t) = \Omega(\hat{\mathbf{r}}(t), R_n). \quad (54)$$

In what follows we often make use of the following elementary identity

$$\int_{\Omega_n} \partial_t f(t, \mathbf{x}) \, d^3x = \partial_t \int_{\Omega_n} f(t, \mathbf{x}) \, d^3x - \int_{\partial\Omega_n} f(t, \mathbf{x}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \, d^3x \quad (55)$$

where $\bar{\mathbf{n}}$ is the external normal to $\partial\Omega_n$, $\hat{\mathbf{v}} = \partial_t \hat{\mathbf{r}}$.

Definition 3.1 (concentrating neighborhoods) Concentrating neighborhoods of a trajectory $\hat{\mathbf{r}}(t)$ are defined as a family of tubular domains

$$\hat{\Omega}(\hat{\mathbf{r}}(t), R_n) = \{(t, \mathbf{x}) : |x - \hat{\mathbf{r}}(t)|^2 \leq R_n^2\} \subset [T_-, T_+] \times \mathbb{R}^3 \quad (56)$$

where R_n satisfy contraction condition:

$$R_n \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (57)$$

We will consider the KG equations and their solutions in concentrating neighborhoods of a trajectory $\hat{\mathbf{r}}(t)$. We will make certain regularity assumptions on the behavior of the potentials and solutions restricted to the domain $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ around the trajectory $\hat{\mathbf{r}}(t)$. The assumptions relate the "microscopic" scales a and a_C to the "macroscopic" length scale of the order 1 relevant to the trajectory $\hat{\mathbf{r}}(t)$ and the electric potential $\varphi(t, \mathbf{x})$.

Below we often make use of the following notations

$$\partial_0 = c^{-1} \partial_t, \quad (58)$$

$$\begin{aligned} \nabla_{\mathbf{x}} \varphi &= \nabla \varphi = (\partial_1 \varphi, \partial_2 \varphi, \partial_3 \varphi), \quad |\nabla_{\mathbf{x}} \varphi|^2 = |\nabla \varphi|^2 = |\partial_1 \varphi|^2 + |\partial_2 \varphi|^2 + |\partial_3 \varphi|^2, \\ \nabla_{0,\mathbf{x}} \varphi &= (\partial_0 \varphi, \partial_1 \varphi, \partial_2 \varphi, \partial_3 \varphi), \quad |\nabla_{0,\mathbf{x}} \varphi|^2 = |\partial_0 \varphi|^2 + |\partial_1 \varphi|^2 + |\partial_2 \varphi|^2 + |\partial_3 \varphi|^2. \end{aligned} \quad (59)$$

Definition 3.2 (localized KG equations) Let $\hat{\mathbf{r}}(t)$ be a trajectory with its concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$, and let $\chi = \chi_n$, $a = a_n$, $\varphi = \varphi_n$ be a sequence of parameters defining the KG equation (3). We call a sequence of the KG equations localized in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ if the following conditions are satisfied. First of all, χ and a become vanishingly small

$$a = a_n \rightarrow 0, \quad \chi = \chi_n \rightarrow 0, \quad (60)$$

the ratio $\zeta = a_C/a$ remains bounded,

$$\zeta = \frac{a_C}{a} \leq C, \quad \text{where } a_C = \frac{\chi}{mc}, \quad (61)$$

and the ratio $\theta = R/a$ grows to infinity

$$\theta_n = \frac{R_n}{a_n} \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (62)$$

Second of all, the electric potentials $\varphi_n(t, \mathbf{x})$ are twice continuously differentiable in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ and satisfy the following constraints:

(i) the following uniform in n estimate holds

$$\max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} (|\varphi_n(t, \mathbf{x})| + |\nabla_{0,\mathbf{x}} \varphi_n(t, \mathbf{x})|) \leq C; \quad (63)$$

(ii) the potentials $\varphi_n(t, \mathbf{x})$ locally converge to a linear potential $\varphi_\infty(t, \mathbf{x})$, namely

$$\max_{T_- \leq t \leq T_+, \mathbf{x} \in \Omega_n} (|\varphi(t, \mathbf{x}) - \varphi_\infty(t, \mathbf{x})| + |\nabla_{0,\mathbf{x}} \varphi(t, \mathbf{x}) - \nabla_{0,\mathbf{x}} \varphi_\infty(t, \mathbf{x})|) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (64)$$

where

$$\varphi_\infty(t, \mathbf{x}) = \varphi_\infty(t, \hat{\mathbf{r}}) + (\mathbf{x} - \hat{\mathbf{r}}) \nabla \varphi_\infty(t), \quad (65)$$

and the coefficients $\varphi_\infty(t, \hat{\mathbf{r}})$ and $\nabla \varphi_\infty(t)$ are bounded

$$|\varphi_\infty(t, \hat{\mathbf{r}})|, \quad |\nabla \varphi_\infty|, \quad |\partial_t \nabla \varphi_\infty| \leq C \quad \text{for } T_- \leq t \leq T_+. \quad (66)$$

Since the parameters m, c, q are fixed, the following inequalities always hold

$$C_0 \chi \leq a_C \leq C_1 \chi, \quad \chi_n \leq C_2 a_n. \quad (67)$$

Based on the above inequalities we often replace χ with a_C in estimates without special comments. Throughout the paper we denote constants which do not depend on χ_n , a_n , and θ_n by the letter C with different indices. Sometimes the same letter C with the same indices may denote in different formulas different constants. Below we often omit index n in χ_n , a_n , φ_n etc.

Obviously, if the potentials φ_n are the restrictions of a fixed twice continuously differentiable function φ to $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$ then conditions (63) and (64) are satisfied with $\varphi_\infty(t, \mathbf{x}) = \varphi(t, \hat{\mathbf{r}}) + (\mathbf{x} - \hat{\mathbf{r}}) \nabla \varphi(t, \hat{\mathbf{r}})$.

Definition 3.3 (concentrating solutions) Let $\hat{\mathbf{r}}(t)$ be a trajectory. We say that solutions ψ of the KG equation (3) concentrate at the trajectory $\hat{\mathbf{r}}(t)$ if the following conditions hold. First of all, a sequence of concentrating neighborhoods $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ and parameters $a = a_n$, $\chi = \chi_n$, potentials $\varphi = \varphi_n$, are selected to form a sequence of the KG equations (3) localized in $\hat{\Omega}_n(\hat{\mathbf{r}}, R_n)$ as in Definition 3.2. Second of all, for every domain $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ there exists a function $\psi = \psi_n \in C^2(\hat{\Omega}(\hat{\mathbf{r}}, R_n))$ which is a solution to the KG equation (3) in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$, and this solution satisfies the following constraints:

- (i) there exists a constant C which does not depend on n but may depend on the sequence such that

$$\max_{T_- \leq t \leq T_+} \int_{\Omega_n} [a_C^2 |\nabla_{0,\mathbf{x}} \psi(t, \mathbf{x})|^2 + a_C^2 |G(|\psi(t, \mathbf{x})|^2)| + |\psi(t, \mathbf{x})|^2] d^3x \leq C, \quad (68)$$

where $G = G_a$ is the nonlinearity in (22);

- (ii) the restriction of functions ψ to the boundary $\partial\Omega_n = \{|\mathbf{x} - \hat{\mathbf{r}}| = R_n\}$ vanishes asymptotically:

$$\max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} (a_C^2 |\nabla_{0,\mathbf{x}} \psi(t, \mathbf{x})|^2 + a_C^2 |G(|\psi(t, \mathbf{x})|^2)| + |\psi(t, \mathbf{x})|^2) d^2\sigma \rightarrow 0; \quad (69)$$

- (iii) the restricted to $\Omega(\hat{\mathbf{r}}(t), R_n)$ energy

$$\bar{\mathcal{E}}_n(t) = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathcal{E}_n d^3x, \quad (70)$$

with \mathcal{E}_n being the energy density defined by expression (26), is bounded from below for sufficiently large n :

$$\bar{\mathcal{E}}_n(t) \geq c_0 > 0 \text{ for } n \geq n_0, \quad T_- \leq t \leq T_+; \quad (71)$$

- (iv) there exists $t_0 \in [T_-, T_+]$ such that the sequence of restricted energies $\bar{\mathcal{E}}_n(t_0)$ converges:

$$\lim_{n \rightarrow \infty} \bar{\mathcal{E}}_n(t_0) = \bar{\mathcal{E}}_\infty(t_0). \quad (72)$$

Notice that condition (i) provides for the boundedness of the restricted energy over domains Ω_n . Condition (ii) provides for a proper confinement of ψ to Ω_n and condition (iii) ensures that the sequence is non-trivial. According to (86) and (68), $\bar{\mathcal{E}}_n(t_0)$ is a bounded sequence, and, consequently, it always contains a converging subsequence. Hence, condition (iv) is not really an additional constraint but rather it assumes that such a subsequence is selected. The choice of a particular subsequence limit $\bar{\mathcal{E}}_\infty(t_0)$ then can be interpreted as a normalization, see Remark 3.9. Obviously, this condition describes the amount of energy which concentrates at the trajectory at the time t_0 .

Example 3.4 The conditions of the Definition 3.3 can be easily verified for rather general sequences of functions which are localized around an arbitrary trajectory $\hat{\mathbf{r}}(t)$ (if we do not

assume that the functions are solutions of the KG equation). In particular, let a sequence of functions be defined by the formula

$$\psi(t, \mathbf{x}) = a^{-3/2} \psi_0((\mathbf{x} - \hat{\mathbf{r}}(t))/a), \quad a = a_n, \quad (73)$$

a is the same parameter as in G_a , the function $\psi_0(z)$ is a smooth fixed function which decays together with its derivatives as $|\mathbf{z}| \rightarrow \infty$:

$$\max_{|z|=\theta} (|\nabla_z \psi_0(\mathbf{z})|^2 + |\psi_0(\mathbf{z})|^2) \leq Y^2(\theta), \quad (74)$$

where $Y(\theta)$ is a continuous function which decays fast enough:

$$Y(\theta) \leq C_0 \theta^{-N} \quad \text{if } \theta \geq 1; \quad Y(\theta) \leq C_0 \quad \text{if } \theta \leq 1, \quad (75)$$

with sufficiently large $N > 3/2$. We assume that N and the sequences $\theta_n \rightarrow \infty$, $a = a_n \rightarrow 0$ satisfy the condition

$$3 - (1 + \alpha)N < 0, \quad a^{-1} \theta_n^{2-(1+\alpha)N} \rightarrow 0 \quad (76)$$

with $1 > \alpha > 0$, α close to 1 as in (12). To verify (68) we change variables $(\mathbf{x} - \hat{\mathbf{r}}(t))/a = \mathbf{z}$, take into account that according to (61) $\chi^2/a^2 \leq C$, and obtain inequalities

$$\begin{aligned} a_C^2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\nabla_x \psi(t, \mathbf{x})|^2 + \frac{1}{c^2} |\partial_t \psi(t, \mathbf{x})|^2 d^3x &\leq C \int_{\mathbf{z} \in \Omega(0, \theta_n)} |\nabla_z \psi_0(t, \mathbf{z})|^2 (1 + |\partial_t \hat{\mathbf{r}}|^2) d^3z \leq \\ &\leq C_1 + C_1 \int_1^\infty r^2 r^{-2N} dr \leq C'_1, \end{aligned}$$

$$\int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\psi(t, \mathbf{x})|^2 d^3x = \int_{\mathbf{z} \in \Omega(0, \theta_n)} |\psi_0(t, \mathbf{z})|^2 d^3z \leq C_2.$$

Now we estimate the integral which involves the nonlinearity

$$G_a(|\psi(t, \mathbf{x})|^2) = a^{-5} G_1(|\psi_0((\mathbf{x} - \hat{\mathbf{r}}(t))/a)|^2) = a^{-5} G_1(|\psi_0(\mathbf{z})|^2).$$

According to (12) and (76)

$$\begin{aligned} a_C^2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |G_a(|\psi(t, \mathbf{x})|^2)| d^3x &= \zeta^2 \int_{\Omega(0, \theta_n)} |G_a(|\psi_0(t, \mathbf{z})|^2)| d^3z \\ &\leq C \zeta^2 \int_{\Omega(0, \theta_n)} |\psi_0(t, \mathbf{z})|^{1+\alpha} d^3z \leq \zeta^2 C_1 + \zeta^2 C_2 \int_1^{\theta_n} \theta^{-N(1+\alpha)+2} d\theta \leq \zeta^2 C_3. \end{aligned} \quad (77)$$

Hence condition (68) is fulfilled. To verify (69) we estimate integrals over the boundary using (76) and (12):

$$\begin{aligned} &\int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} a_C^2 |\nabla_x \psi(t, \mathbf{x})|^2 + a_C^2 \frac{1}{c^2} |\partial_t \psi(t, \mathbf{x})|^2 + |\psi(t, \mathbf{x})|^2 d^2\sigma \\ &\leq C a^{-1} \int_{\partial\Omega(0, \theta_n)} |\nabla_z \psi_0(t, \mathbf{z})|^2 + |\psi_0(t, \mathbf{z})|^2 d^2\sigma \leq C_1 \theta_n^2 Y^2(\theta_n) a^{-1} \leq C, \\ &a_C^2 \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} |G_a(|\psi(t, \mathbf{x})|^2)| d^2\sigma = \zeta^2 a^{-1} \int_{\partial\Omega(0, \theta_n)} |G_1(|\psi_0(t, \mathbf{z})|^2)| d^2\sigma \\ &\leq C \zeta^2 a^{-1} \int_{\partial\Omega(0, \theta_n)} |\psi_0(t, \mathbf{z})|^{1+\alpha} d^2\sigma \leq C_1 \zeta^2 a^{-1} \theta_n^2 Y^{1+\alpha}(\theta_n) \leq C_2 a^{-1} \theta_n^2 \theta_n^{-N(1+\alpha)} \leq C'_2. \end{aligned}$$

Consequently, we obtain the desired (69). The energy $\bar{\mathcal{E}}_n$ involves the positive term

$$\frac{\chi^2}{2m} \kappa_0^2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\psi(t, \mathbf{x})|^2 = \frac{mc^2}{2} \int_{\Omega(0, \theta_n)} |\psi_0|^2 d\mathbf{z} \rightarrow \frac{mc^2}{2} \int_{\mathbb{R}^3} |\psi_0|^2 d\mathbf{z} \geq \frac{1}{C_4} > 0.$$

According to (26), the only negative contribution to the energy can come from the nonlinearity G , and, according to (77), this contribution cannot be greater than $\zeta^2 C_3$, hence condition (71) is satisfied if ζ is small enough.

In the following important example we consider uniform motion of charges from Section 2.4 where the relativistic argument is used to determine their mass. The uniformly moving solutions are also an example of concentrating solutions.

Example 3.5 *As a simple example of solutions which concentrate at a trajectory $\hat{\mathbf{r}}(t)$ we take solutions defined by (43) in Section 2.4. Then the trajectory $\hat{\mathbf{r}}(t) = \mathbf{v}t$ is a straight line, $\varphi = 0$. We assume that the function $\psi_1 = \psi_0$ from (43) satisfies (74), (75) (ground states from Examples 2.1-2.3 satisfy this assumption). We take sequences $a_n \rightarrow 0$, $\chi_n \rightarrow 0$, $R_n \rightarrow 0$, $\theta_n \rightarrow \infty$ such that conditions (76) are satisfied. Since $\theta_n \rightarrow \infty$, the restricted energy and charge defined as integrals over $\Omega(\hat{\mathbf{r}}(t), R_n)$ converge to the integrals over the entire space and $\bar{\mathcal{E}}_\infty, \bar{\rho}_\infty$ are given by (47), namely*

$$\bar{\mathcal{E}}_n(t) \rightarrow \gamma mc^2 (1 + \Theta(\omega)), \quad \bar{\rho}_n(t) \rightarrow q.$$

Therefore (71) holds for large n . The solutions concentrate at $\hat{\mathbf{r}}(t) = \mathbf{v}t$ and have the following additional properties: (i) the energy density \mathcal{E} is center-symmetric with respect to $\hat{\mathbf{r}}(t) = \mathbf{v}t$, hence the ergocenter $\mathbf{r}(t)$ defined by (99) coincides with the center $\hat{\mathbf{r}}(t)$; (ii) the charge density ρ given by (47) and according to (44) it converges to delta-function $q\delta(\mathbf{x} - \mathbf{r})$ as $a \rightarrow 0$; (iii) the current \mathbf{J} is given by (48), its components are center-symmetric and converge to the corresponding components of $q\mathbf{v}\delta(\mathbf{x} - \mathbf{r})$.

The following statement is the main result of this paper. It describes trajectories relevant to concentrating solutions to the KG equations.

Theorem 3.6 (relevant trajectories) *Let solutions ψ of the KG equation (3) concentrate at $\hat{\mathbf{r}}(t)$. Then the restricted energies $\bar{\mathcal{E}}_n(t)$ converge to the limit restricted energy $\bar{\mathcal{E}}_\infty(t)$ which satisfies equation (91). The limit energy $\bar{\mathcal{E}}_\infty(t)$ and the trajectory $\hat{\mathbf{r}}(t)$ satisfy equation*

$$\partial_t \left(\frac{1}{c^2} \bar{\mathcal{E}}_\infty(t) \partial_t \hat{\mathbf{r}} \right) = \mathbf{f}_\infty \quad (78)$$

with the electric force $\mathbf{f}_\infty(t)$ given by the formula

$$\mathbf{f}_\infty(t) = -\bar{\rho}_\infty \nabla \varphi_\infty(t) \quad (79)$$

where the charge $\bar{\rho}_\infty$ does not depend on t . If we identify the coefficient at $\partial_t \hat{\mathbf{r}}$ in (78) with the mass M by Einstein's formula

$$M = \frac{1}{c^2} \bar{\mathcal{E}}_\infty(t), \quad (80)$$

the following formula holds:

$$M = \gamma M_0, \quad \gamma = (1 - (\partial_t \hat{\mathbf{r}})^2 / c^2)^{-1/2} \quad (81)$$

where M_0 is a constant.

The proof of Theorem 3.6 is given in Section 3.3.

Note that the formula (80) defines the mass M based on the requirement that (78) takes the form of the relativistic version of Newton's equation (2). Therefore formula (78) provides an alternative Newtonian method of deriving Einstein's formula.

Remark 3.7 *The constant M_0 in (81) equals the mass M if $\partial_t \hat{\mathbf{r}} = \mathbf{0}$. This allows to interpret M_0 as the rest mass as in (2). We would like to stress that the rest mass M_0 in our treatment is not a prescribed quantity, but it is derived in (121) as an integral of motion of an equation obtained in asymptotic limit from the KG equation. As any integral of motion it can take different values for different "trajectories" of the field. Notice that the rest mass M_0 can take different values for different rest states described in Section 2.3. The integral of motion M_0 can be related to the mass m_0 of one of resting charges considered in Section 2.4 by the identity*

$$M_0 = m_0 \quad (82)$$

if the velocity vanishes on a time interval or asymptotically as $t \rightarrow -\infty$ or $t \rightarrow \infty$. If the velocity $\partial_t \hat{\mathbf{r}}$ vanishes just at a time instant t_0 then it is, of course, possible to express the value of M_0 in terms of $\bar{\mathcal{E}}_\infty = \bar{\mathcal{E}}_\infty(\psi)$ by formulas (80), (121), but the corresponding $\psi = \psi(t_0)$ may have no relation to the rest solutions of the field equation with a time independent profile $|\psi|^2$. It is also possible that $\partial_t \hat{\mathbf{r}}$ never equals zero, and in fact this is a general case since all three components of velocity may vanish simultaneously only in very special situations. Hence, there is a possibility of localized regimes where the value of the "rest mass" M_0 may differ from the rest mass of a free charge. In such regimes the value of the rest mass cannot be derived based on the analysis of the uniform motion as in Section 2.4. This wide variety of possibilities makes even more remarkable the fact that the inertial mass is well-defined and that Einstein's formula (1) holds even in such general regimes where the standard analysis based on the Lorentz invariance of the uniform motion as in Section 2.4 does not apply.

Remark 3.8 *If the external electric field φ is not zero, the system described by (22) is not a closed system, and there are principal differences between closed and non-closed systems. In particular, the total momentum and the energy of a closed system are preserved and form a 4-vector. For non-closed systems the center of energy (also known as center of mass or centroid) and the total energy-momentum are frame dependent and hence are not 4-vectors, [27, Sec. 7.1, 7.2], [24, Sec. 24]. Hence, one cannot identify the velocity parameter \mathbf{v} of the Lorentz group with the velocity of the system.*

Remark 3.9 *The sequences $a_n, R_n, \theta_n, \varphi_n, \chi_n, \psi_n$ enter the definition of a concentrating solution. But if we take two different sequences which fit the definition for the same trajectory, we obtain the same equation (78)-(81) and the limit energy $E_\infty(t)$. More than that, as long as the gradient $\nabla \varphi_\infty(t)$ of the limit potential is given and at a moment of time t_0 the position and velocity $\hat{\mathbf{r}}(t_0), \partial_t \hat{\mathbf{r}}(t_0)$ and the limit restricted energy and charge $E_\infty(t_0), \bar{\rho}_\infty$ are fixed, then the trajectory $\hat{\mathbf{r}}(t)$ and the energy $E_\infty(t)$ are uniquely defined as solutions of equations (78)-(81), and consequently they do not depend on the particular sequences. Note also that if in Definition 3.3 we would eliminate point (iv), we could pick a subsequence for which (72) holds. For any such a subsequence we obtain (78), but $\bar{\mathcal{E}}_\infty(t_0)$ could be different. Taking into account (81) and Corollary 3.16 we see that the choice of different subsequences leads to multiplication of equations (78), (81) by a constant and corresponds to simultaneous multiplication of the rest mass M_0 and the charge $\bar{\rho}_\infty$ by the same constant.*

In Section 4 we construct a non-trivial example of sequences of the KG equations and their solutions which concentrate at rectilinear accelerating trajectories.

3.2 Properties of concentrating solutions

Everywhere in this section we assume that we have a trajectory $\hat{\mathbf{r}}(t)$ with its concentrating neighborhoods $\hat{\Omega}_n(\hat{\mathbf{r}}, R_n)$. In this section we often use the following elementary inequalities for the densities defined in Section 2.2:

$$|\mathbf{P}| \leq \frac{\chi^2}{mc^2} \left| \partial_t \psi + \frac{iq}{\chi} \varphi \psi \right| |\nabla \psi| \leq C\chi^2 |\partial_t \psi|^2 + C\chi^2 |\nabla \psi|^2 + C|\psi|^2, \quad (83)$$

$$|\mathbf{J}| \leq \frac{\chi q}{m} |\nabla \psi| |\psi| \leq C(\chi |\nabla \psi| + |\psi|) |\psi| \leq C'\chi^2 |\nabla \psi|^2 + C'|\psi|^2, \quad (84)$$

$$|\rho| \leq \frac{\chi q}{mc^2} \left| \tilde{\partial}_t \psi \right| |\psi| \leq C\chi^2 |\partial_t \psi|^2 + C|\psi|^2. \quad (85)$$

Similarly,

$$|\mathcal{E}| + |\mathcal{L}| \leq C\chi^2 |\partial_t \psi|^2 + C\chi^2 |\nabla \psi|^2 + C\chi^2 |G(\psi^* \psi)| + C|\psi|^2. \quad (86)$$

We define the restricted to $\Omega(\hat{\mathbf{r}}(t), R_n)$ charge $\bar{\rho}_n$ by the formula

$$\bar{\rho}_n(t) = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \rho_n d^3x. \quad (87)$$

Lemma 3.10 *Let solutions ψ of the KG equation (3) concentrate at $\hat{\mathbf{r}}(t)$. Then there exists a number $\bar{\rho}_\infty$ and a subsequence of the solutions ψ_n such that*

$$\bar{\rho}_n(t) \rightarrow \bar{\rho}_\infty \quad \text{uniformly for } T_- \leq t \leq T_+. \quad (88)$$

Proof. According to (85) and (68), for any $t_0 \in [T_-, T_+]$ $\bar{\rho}_n(t_0)$ is a bounded sequence, and hence it always contains a converging subsequence. We pick such a subsequence, denote its limit by $\bar{\rho}_\infty$ and integrate the continuity equation (30):

$$\partial_t \int_{\Omega_n} \rho_n d^3x - \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho_n d^2\sigma + \int_{\partial\Omega_n} \bar{\mathbf{n}} \cdot \mathbf{J} d^2\sigma = 0. \quad (89)$$

Integrating with respect to time from t_0 to t and using (85), (84) and (69), we obtain

$$\begin{aligned} |\bar{\rho}_n(t) - \bar{\rho}_\infty| &\leq |\bar{\rho}_n(t) - \bar{\rho}_n(t_0)| + |\bar{\rho}_n(t_0) - \bar{\rho}_\infty| \leq (T_+ - T_-) |\hat{\mathbf{v}}| \int_{\partial\Omega_n} |\rho_n| d^2\sigma \\ &+ (T_+ - T_-) \int_{\partial\Omega_n} |\mathbf{J}| d^2\sigma \leq C \int_{\partial\Omega_n} (\chi^2 |\nabla_{0,x} \psi|^2 + C'_1 |\psi|^2) d^2\sigma + |\bar{\rho}_n(t_0) - \bar{\rho}_\infty| \rightarrow 0. \end{aligned}$$

Therefore (88) holds. ■

Theorem 3.11 (restricted energy convergence) *Let solutions ψ of the KG equation (3) concentrate at $\hat{\mathbf{r}}(t)$. Let $\bar{\rho}_n(t_0) \rightarrow \bar{\rho}_\infty$. Then the restricted energy $\bar{\mathcal{E}}_n(t)$ converges uniformly to the limit restricted energy $\bar{\mathcal{E}}_\infty$*

$$\bar{\mathcal{E}}_n(t) \rightarrow \bar{\mathcal{E}}_\infty(t) \quad \text{uniformly on } [T_-, T_+], \quad (90)$$

where

$$\bar{\mathcal{E}}_\infty(t) = \bar{\mathcal{E}}_\infty(t_0) - \bar{\rho}_\infty \int_{t_0}^t \partial_t \hat{\mathbf{r}} \cdot \nabla \varphi_\infty dt'. \quad (91)$$

Proof. Integrating (32) with respect to x and t we obtain

$$\bar{\mathcal{E}}_n(t) - \bar{\mathcal{E}}_\infty(t_0) = - \int_{t_0}^t \nabla \varphi_\infty \cdot \int_{\Omega_n} \mathbf{J} d^3x dt' + Q_1 \quad (92)$$

where φ_∞ is defined by (65), and

$$Q_1 = \int_{t_0}^t \int_{\partial\Omega_n} (\hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \mathcal{E}_n - c^2 \bar{\mathbf{n}} \cdot \mathbf{P}) d^2\sigma dt' - \int_{t_0}^t \int_{\Omega_n} \nabla (\varphi - \varphi_\infty) \cdot \mathbf{J} d^3x dt' + \bar{\mathcal{E}}_n(t_0) - \bar{\mathcal{E}}_\infty(t_0).$$

We denote $\varphi - \varphi_\infty = \varphi_b$, and, using the continuity equation (30), we obtain

$$\int_{\Omega_n} \nabla \varphi_b \cdot \mathbf{J} d^3x = \partial_t \int_{\Omega_n} \varphi_b \rho d^3x + \int_{\partial\Omega_n} (\bar{\mathbf{n}} \cdot \mathbf{J} - \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho) \varphi_b d^2\sigma - \int_{\Omega_n} \partial_t \varphi_b \rho d^3x.$$

Therefore, according to (64), (85), (84), (68) and (69)

$$\begin{aligned} \left| \int_{t_0}^t \int_{\Omega_n} \nabla \varphi_b \cdot \mathbf{J} d^3x \right| &\leq 2 \max_{T_- \leq t \leq T_+, x \in \Omega_n} |\varphi_b| \max_{T_- \leq t \leq T_+} \int_{\Omega_n} |\rho| d^3x \\ + C \max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} (|\mathbf{J}| + |\rho|) \varphi_b d^2\sigma &+ \max_{T_- \leq t \leq T_+, x \in \Omega_n} |\partial_t \varphi_b| \max_{T_- \leq t \leq T_+} \int_{\Omega_n} |\rho| d^3x \rightarrow 0. \end{aligned} \quad (93)$$

Similarly, we estimate the first term in Q_1

$$\left| \int_{t_0}^t \int_{\partial\Omega_n} \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \mathcal{E}_n - c^2 \bar{\mathbf{n}} \cdot \mathbf{P} d^2\sigma dt' \right| \leq C \max_{T_- \leq t \leq T_+} \int_{\partial\Omega_n} (|\mathcal{E}_n| + |\mathbf{P}|) d^2\sigma dt'$$

and conclude that $|Q_1| \rightarrow 0$.

Multiplication of the continuity equation (30) by the vector $\mathbf{x} - \hat{\mathbf{r}}$ produces the following expression for \mathbf{J} :

$$\partial_t (\rho (\mathbf{x} - \hat{\mathbf{r}})) + \partial_t \hat{\mathbf{r}} \rho + \sum_j \nabla_j ((\mathbf{x} - \hat{\mathbf{r}}) \mathbf{J}_j) = \mathbf{J}. \quad (94)$$

Integrating we see that

$$\int_{\Omega_n} \mathbf{J} d^3x = \hat{\mathbf{v}} \bar{\rho} + \partial_t \int_{\Omega_n} (\rho (\mathbf{x} - \hat{\mathbf{r}})) d^3x + \int_{\partial\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) (\bar{\mathbf{n}} \cdot \mathbf{J} - \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho) d^2\sigma$$

and

$$\int_{t_0}^t \nabla \varphi_\infty \cdot \int_{\Omega_n} \mathbf{J} d^3x dt' = \int_{t_0}^t \nabla \varphi_\infty \cdot \partial_t \hat{\mathbf{r}} \bar{\rho}_\infty dt' + Q_{10} \quad (95)$$

where

$$Q_{10} = \int_{\Omega_n} (\rho (\mathbf{x} - \hat{\mathbf{r}})) d^3x \Big|_{t_0}^t + \int_{t_0}^t \int_{\partial\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) (\bar{\mathbf{n}} \cdot \mathbf{J} - \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho) d^2\sigma dt' + \int_{t_0}^t \nabla \varphi_\infty \cdot \partial_t \hat{\mathbf{r}} (\bar{\rho} - \bar{\rho}_\infty) dt'$$

Combining with (92), we obtain

$$\bar{\mathcal{E}}_n(t) - \bar{\mathcal{E}}_\infty(t_0) = - \int_{t_0}^t \nabla \varphi_\infty \cdot \partial_t \hat{\mathbf{r}}(t) \bar{\rho}_\infty dt' + Q_1 - Q_{10} \quad (96)$$

Using (66), (85), (84), (52), (68), we conclude that

$$\begin{aligned} |Q_{10}| &\leq 2R_n \sup_{T_- \leq t \leq T_+} \int_{\Omega_n} |\rho| \, d^3x + (T_+ - T_-) R_n \sup_{T_- \leq t \leq T_+} |\hat{\mathbf{v}}| \int_{\partial\Omega_n} |\rho| \, d^2\sigma \\ &+ (T_+ - T_-) R_n \int_{\partial\Omega_n} |\mathbf{J}| \, d^2\sigma + (T_+ - T_-) \sup_{T_- \leq t \leq T_+} |\nabla\varphi_\infty| \cdot \hat{\mathbf{v}} |\bar{\rho} - \bar{\rho}_\infty| \rightarrow 0. \end{aligned} \quad (97)$$

We obtain (90) and (91) from (96). ■

Remark 3.12 *Equations (2) for the space components of the relativistic 4-vector are usually complemented (see, for instance, [8], [30], [27]) with the time component*

$$\frac{d}{dt} (Mc^2) = \mathbf{f} \cdot \mathbf{v}. \quad (98)$$

Formula (91), obviously, has the form of integrated equation (98).

We define the adjacent ergocenter \mathbf{r}_n by the formula

$$\mathbf{r}_n(t) = \frac{1}{\mathcal{E}_n} \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathbf{x} \mathcal{E}_n \, d^3x. \quad (99)$$

Lemma 3.13 *Let solutions of the KG equation (3) concentrate at $\hat{\mathbf{r}}(t)$. Then the adjacent ergocenters $\mathbf{r}_n(t)$ of the solutions converge to $\hat{\mathbf{r}}(t)$ uniformly on the time interval $[T_-, T_+]$.*

Proof. We infer from (70) and (99) that

$$\frac{1}{\mathcal{E}_n} \int_{\Omega_n} (\mathbf{x} - \mathbf{r}_n) \mathcal{E}_n \, d^3x = \mathbf{r}_n - \mathbf{r}_n \frac{1}{\mathcal{E}_n} \int_{\Omega_n} \mathcal{E}_n \, d^3x = 0. \quad (100)$$

From (68) we obtain

$$\left| \int_{\Omega_n} (\mathbf{x} - \hat{\mathbf{r}}) \mathcal{E}_n \, d^3x \right| \leq R_n \int_{\Omega_n} |\mathcal{E}_n| \, d^3x \leq CR_n \rightarrow 0. \quad (101)$$

Using (71) we conclude that

$$|\hat{\mathbf{r}} - \mathbf{r}_n| = \frac{1}{\mathcal{E}_n} \left| \int_{\Omega_n} (\mathbf{x} - \mathbf{r}_n) - (\mathbf{x} - \hat{\mathbf{r}}) \mathcal{E}_n \, d^3x \right| \leq C_1 R_n \rightarrow 0. \quad (102)$$

■

Lemma 3.14 *Let solutions $\psi = \psi_n$ of the KG equation (3) concentrate at $\hat{\mathbf{r}}(t)$. Then*

$$|\mathbf{v}_n| = |\partial_t \mathbf{r}_n| \leq C_4 \quad (103)$$

and for any t_0 there exists a subsequence such that $\mathbf{v}_n(t_0)$ converges.

Proof. Multiplying (32) by $(\mathbf{x} - \mathbf{r})$, we obtain

$$\partial_t ((\mathbf{x} - \mathbf{r}) \mathcal{E}) + \mathcal{E} \partial_t \mathbf{r} = -(\mathbf{x} - \mathbf{r}) c^2 \nabla \cdot \mathbf{P} - (\mathbf{x} - \mathbf{r}) \nabla \varphi \cdot \mathbf{J}. \quad (104)$$

Integration over Ω_n yields

$$\int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \mathcal{E}) d^3x + \partial_t \mathbf{r} \int_{\Omega_n} \mathcal{E} d^3x = - \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) c^2 \nabla \cdot \mathbf{P} d^3x - \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla \varphi \cdot \mathbf{J} d^3x.$$

From the definition of the ergocenter \mathbf{r} we infer that

$$\int_{\Omega_n} \partial_t ((\mathbf{x} - \mathbf{r}) \mathcal{E}) d^3x + \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \mathcal{E} d^2\sigma = 0. \quad (105)$$

Therefore

$$\partial_t \mathbf{r} \mathcal{E}_n = \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \mathcal{E} d^2\sigma - c^2 \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla \cdot \mathbf{P} d^3x - \int_{\Omega_n} (\mathbf{x} - \mathbf{r}) \nabla \varphi \cdot \mathbf{J} d^3x.$$

Using (63), (71) and (102) we obtain the inequality

$$|\partial_t \mathbf{r}| \leq C \int_{\partial\Omega_n} (|\mathcal{E}| + |\mathbf{P}|) d^2\sigma + \int_{\Omega_n} (|\mathbf{J}| + |\mathbf{P}|) d^3x$$

yielding (103). ■

3.3 Proof of Theorem 3.6

We prove in this section that if solutions to the KG equation concentrate at a trajectory $\hat{\mathbf{r}}(t)$ then the limit restricted energy and the trajectory must satisfy Einstein's formula and the relativistic version of Newton's law. The proof is based on two facts. First, according to Lemma 3.13 the trajectory $\hat{\mathbf{r}}(t)$ is the limit of adjacent ergocenters $\mathbf{r}_n(t)$. Second, the adjacent ergocenters satisfy equations (see Theorem 3.15) which yield in the limit the relativistic Newton's law and Einstein's formula.

Theorem 3.15 *Let solutions ψ_n of the KG equation (3) concentrate at $\hat{\mathbf{r}}(t)$, let $\mathbf{r}(t) = \mathbf{r}_n(t)$ be adjacent ergocenters defined by (99). Then for any $t_0, t \in [T_-, T_+]$*

$$\frac{1}{c^2} \bar{\mathcal{E}}_n(t) \partial_t \mathbf{r}(t) - \frac{1}{c^2} \bar{\mathcal{E}}_n(t_0) \partial_t \mathbf{r}(t_0) = - \int_{t_0}^t \bar{\rho}_\infty \nabla \varphi_\infty dt' + \boldsymbol{\delta}_f \quad (106)$$

where $\boldsymbol{\delta}_f \rightarrow 0$ uniformly on $[T_-, T_+]$.

Proof. Integrating the momentum conservation equation (33) with respect to time and over $\Omega(\hat{\mathbf{r}}(t), R_n) = \Omega_n$ we obtain

$$\begin{aligned} & \int_{\Omega_n} \mathbf{P}(t) d^3x - \int_{\Omega_n} \mathbf{P}(t_0) d^3x - \int_{\Omega_n} \int_{t_0}^t \mathbf{F} dt' d^3x \\ & + \int_{\Omega_n} \int_{t_0}^t \nabla \mathcal{L}_1(\psi) dt' d^3x + \frac{\chi^2}{2m} \int_{\Omega_n} \int_{t_0}^t \sum_j \partial_j (\partial_j \psi \nabla \psi^* + \partial_j \psi^* \nabla \psi) dt' d^3x = 0. \end{aligned}$$

Since $\Omega_n = \Omega(\hat{\mathbf{r}}(t), R_n)$ does not depend on t' , we can change order of integration and get

$$\int_{\Omega_n} \mathbf{P}(t) d^3x - \int_{\Omega_n} \mathbf{P}(t_0) d^3x + Q_0 = \int_{t_0}^t \int_{\Omega_n} \mathbf{F} dt', \quad (107)$$

$$Q_0 = \int_{t_0}^t \int_{\partial\Omega_n} \bar{\mathbf{n}} \mathcal{L}_1(\psi) d^2\sigma dt' + \frac{\chi^2}{2m} \int_{t_0}^t \int_{\partial\Omega_n} (\bar{\mathbf{n}} \cdot \nabla \psi \nabla \psi^* + \bar{\mathbf{n}} \cdot \nabla \psi^* \nabla \psi) d^2\sigma dt'. \quad (108)$$

Using (86) and (69), we infer that

$$|Q_0| \leq |t - t_0| \max_{T_- \leq s \leq T_+} \int_{\partial\Omega_n} (|\mathcal{L}_1(\psi)| + \chi^2 m^{-1} |\nabla \psi|^2) d^2\sigma \rightarrow 0. \quad (109)$$

Note that

$$\int_{\Omega_n} \mathbf{P}(t_0) d^3x = \int_{\Omega(\hat{\mathbf{r}}(t_0), R_n)} \mathbf{P}(t_0) d^3x + Q_{01}, \quad (110)$$

$$Q_{01} = \int_{\Omega(\mathbf{0}, R_n)} \mathbf{P}(t_0, \mathbf{y} + \hat{\mathbf{r}}(t)) - \mathbf{P}(t_0, \mathbf{y} + \hat{\mathbf{r}}(t_0)) d^3y. \quad (111)$$

Obviously

$$\begin{aligned} Q_{01} &= \int_{\Omega(\mathbf{0}, R_n)} \int_{t_0}^t \partial_s \mathbf{P}(t_0, \mathbf{y} + \hat{\mathbf{r}}(s)) ds d^3y = \int_{t_0}^t \int_{\Omega(\mathbf{0}, R_n)} \partial_t \hat{\mathbf{r}}(s) \cdot \nabla \mathbf{P}(t_0, \mathbf{y} + \hat{\mathbf{r}}(s)) d^3y ds \\ &= \int_{t_0}^t \int_{\partial\Omega(\mathbf{0}, R_n)} \bar{\mathbf{n}} \cdot \hat{\mathbf{v}}(s) \mathbf{P}(t_0, \mathbf{y} + \hat{\mathbf{r}}(s)) d^2\sigma ds. \end{aligned}$$

Using (83) and (69) we infer that

$$|Q_{01}| \leq |t - t_0| \max_{T_- \leq s \leq T_+} \int_{\partial\Omega_n} |\mathbf{P}| d^2\sigma \rightarrow 0.$$

We obtain from the relation (104) the following expression for \mathbf{P} :

$$\mathbf{P} = \frac{1}{c^2} \partial_t ((\mathbf{x} - \mathbf{r}) \mathcal{E}) + \frac{1}{c^2} \mathcal{E} \partial_t \mathbf{r} + \sum_j \partial_j \cdot (\mathbf{P}_j (\mathbf{x} - \mathbf{r})) + \frac{1}{c^2} (\nabla \varphi \cdot \mathbf{J}) (\mathbf{x} - \mathbf{r}). \quad (112)$$

We use this expression to write the first term in (107) as follows:

$$\int_{\Omega_n} \mathbf{P}(t) d^3x = \frac{1}{c^2} \partial_t \mathbf{r} \int_{\Omega_n} \mathcal{E} d^3x + Q_{02} \quad (113)$$

where

$$Q_{02} = \int_{\Omega_n} \left(\frac{1}{c^2} \partial_t ((\mathbf{x} - \mathbf{r}) \mathcal{E}) + \sum_j \partial_j \cdot (\mathbf{P}_j (\mathbf{x} - \mathbf{r})) + \frac{1}{c^2} (\nabla \varphi \cdot \mathbf{J}) (\mathbf{x} - \mathbf{r}) \right) d^3x. \quad (114)$$

Using (105) we obtain

$$Q_{02} = -\frac{1}{c^2} \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \mathcal{E} d^3x + \int_{\partial\Omega_n} (\mathbf{x} - \mathbf{r}) \bar{\mathbf{n}} \cdot \mathbf{P} d^3x + \int_{\Omega_n} \frac{1}{c^2} (\nabla \varphi \cdot \mathbf{J}) (\mathbf{x} - \mathbf{r}) d^3x.$$

Using (102), (83), (86) and (69) we conclude that

$$|Q_{02}| \leq CR_n \int_{\partial\Omega_n} |\mathcal{E}| \, d^3x + CR_n \int_{\partial\Omega_n} |\mathbf{P}| \, d^3x + CR_n \max_{\Omega_n} |\nabla\varphi| \int_{\Omega_n} |\mathbf{J}| \, d^3x \rightarrow 0.$$

Quite similarly to (113),

$$\int_{\Omega(\hat{\mathbf{r}}(t_0), R_n)} \mathbf{P}(t_0) \, d^3x = \frac{1}{c^2} \partial_t \mathbf{r}(t_0) \bar{\mathcal{E}}(t_0) + Q_{012} \quad (115)$$

where $Q_{012} \rightarrow 0$. Now we write the last term in (107) in the form

$$\begin{aligned} \int_{t_0}^t \int_{\Omega_n} \mathbf{F} \, d^3x dt' &= - \int_{t_0}^t \nabla\varphi_\infty \int_{\Omega_n} \rho_\infty \, d^3x dt' + Q_{03}, \\ Q_{03} &= - \int_{t_0}^t \int_{\Omega_n} (\rho_n \nabla\varphi - \rho_\infty \nabla\varphi_\infty) \, d^3x. \end{aligned} \quad (116)$$

Obviously

$$\int_{\Omega_n} (\rho_n \nabla\varphi - \rho_\infty \nabla\varphi_\infty) \, d^3x = \int_{\Omega_n} (\nabla\varphi - \nabla\varphi_\infty) \rho_n \, d^3x + \nabla\varphi_\infty (\bar{\rho}_n - \bar{\rho}_\infty),$$

and by Lemma 3.10 and (64)

$$|Q_{03}| \leq |T_+ - T_-| \max_{\Omega_n} |\nabla\varphi - \nabla\varphi_\infty| \int_{\Omega_n} |\rho_n| \, d^3x + |T_+ - T_-| \max_t |\nabla\varphi_\infty| |\rho_n - \rho_\infty| \rightarrow 0.$$

From (107) using (113), (110), (115), (116) we obtain

$$\frac{1}{c^2} \bar{\mathcal{E}}_n(t) \partial_t \mathbf{r}(t) + Q_{02} - \frac{1}{c^2} \bar{\mathcal{E}}_n(t_0) \partial_t \mathbf{r}(t_0) - Q_{01} - Q_{012} + Q_0 = - \int_{t_0}^t \bar{\rho}_\infty \nabla\varphi_\infty dt' + Q_{03}$$

which implies (106) with $\delta_f = Q_{03} - Q_0 + Q_{012} + Q_{01} - Q_{02}$. ■

Proof of Theorem 3.6. According to Lemma 3.13, the adjacent ergocenters $\mathbf{r}_n(t)$ converge to $\hat{\mathbf{r}}(t)$. We take t_0 from Definition 3.3 and choose such a subsequence that according to Lemma 3.14 and 3.10 $\partial_t \mathbf{r}_n(t_0) \rightarrow v_\infty$, $\bar{\rho}_n \rightarrow \bar{\rho}_\infty$.

We multiply (106) by $c^2/\bar{\mathcal{E}}_n$ and obtain

$$\partial_t \mathbf{r}_n(t) = \frac{c^2}{\bar{\mathcal{E}}_\infty} \int_{t_0}^t \mathbf{f}_\infty dt' + \frac{c^2}{\bar{\mathcal{E}}_\infty} \delta_f + \frac{1}{\bar{\mathcal{E}}_\infty} \bar{\mathcal{E}}_n(t_0) \partial_t \mathbf{r}_n(t_0). \quad (117)$$

Integration of the above equation yields

$$\mathbf{r}_n(t) - \mathbf{r}_n(t_0) = \int_{t_0}^t \frac{c^2}{\bar{\mathcal{E}}_\infty} \delta_f dt'' + \int_{t_0}^t \frac{c^2}{\bar{\mathcal{E}}_\infty} \int_{t_0}^{t''} \mathbf{f}_\infty dt' dt'' + \int_{t_0}^t \frac{1}{\bar{\mathcal{E}}_\infty(t'')} dt'' \bar{\mathcal{E}}_n(t_0) \partial_t \mathbf{r}_n(t_0).$$

Observe that the sequence $1/\bar{\mathcal{E}}_n$ converges uniformly to $1/\bar{\mathcal{E}}_\infty$ according to Lemma 3.11 and (71), $\mathbf{r}_n(t)$ converges to $\hat{\mathbf{r}}(t)$ and δ_f converge to zero. Hence taking the limit of the above we get

$$\hat{\mathbf{r}}(t) - \hat{\mathbf{r}}(t_0) = \int_{t_0}^t \frac{c^2}{\bar{\mathcal{E}}_\infty} \int_{t_0}^{t''} \mathbf{f}_\infty dt' dt'' + \int_{t_0}^t \frac{1}{\bar{\mathcal{E}}_\infty(t'')} dt'' \bar{\mathcal{E}}_\infty(t_0) v_\infty. \quad (118)$$

Taking the time derivative of the equality (118), multiplying the result by $\bar{\mathcal{E}}_\infty(t)/c^2$ and taking the time derivative once more we find that $\hat{\mathbf{r}}(t)$ satisfies (78) and $\hat{\mathbf{r}}(t_0) = \mathbf{r}_\infty(t_0)$, $\partial_t \hat{\mathbf{r}}(t_0) = v_\infty$.

Notice that equation (91) implies

$$\partial_t \bar{\mathcal{E}}_\infty = \partial_t \hat{\mathbf{r}} \cdot \mathbf{f}_\infty. \quad (119)$$

Multiplying (78) by $2M\partial_t \hat{\mathbf{r}}$ where $M = \bar{\mathcal{E}}_\infty(t)/c^2$ and using (119), we obtain

$$\partial_t (M\partial_t \hat{\mathbf{r}})^2 = 2M\partial_t \hat{\mathbf{r}} \cdot \mathbf{f}_\infty = 2c^2 M \partial_t M. \quad (120)$$

This equation after time integration implies

$$M^2 - c^{-2} M^2 (\partial_t \hat{\mathbf{r}})^2 = M_0^2, \quad (121)$$

where M_0^2 is a constant of integration; consequently, we obtain (81). Note that $M_0 = \gamma^{-1}(t_0) c^{-2} \bar{\mathcal{E}}_\infty(t_0)$ is uniquely determined by the value $\bar{\mathcal{E}}_\infty(t_0)$ from (72). Hence the limit of the restricted energy $\bar{\mathcal{E}}_\infty(t)$ does not depend on particular subsequences $\partial_t \mathbf{r}_n(t_0)$, $\bar{\rho}_n$ and on $\bar{\rho}_\infty$. Based on this we conclude that any subsequence $\bar{\mathcal{E}}_n(t)$ has the same limit and the convergence in (90) holds for the given concentrating sequence. ■

Corollary 3.16 *Assume that $\partial_t^2 \hat{\mathbf{r}}$ is not identically zero on $[T_-, T_+]$. Then (88) holds for any subsequence of the concentrating sequence and the sequence $\bar{\rho}_n(t)$ converges to $\bar{\rho}_\infty$.*

Proof. We have to prove that every convergent subsequence $\bar{\rho}_n(t_0)$ in Lemma 3.10 converges to the same limit. If we have a convergent subsequence $\bar{\rho}_n(t_0)$ which converges to $\bar{\rho}_\infty$, we obtain (81) and (78). According to formula (81), we can rewrite (78) in the form

$$M_0 \gamma \partial_t^2 \hat{\mathbf{r}} + M_0 \partial_t \gamma \partial_t \hat{\mathbf{r}} = -\bar{\rho}_\infty \nabla \varphi_\infty. \quad (122)$$

This relation uniquely determines $\bar{\rho}_\infty$ if $\nabla \varphi_\infty(t)$ is not identically zero. The left-hand side vanishes on an interval only if $\partial_t \hat{\mathbf{v}} + \partial_t \ln \gamma \hat{\mathbf{v}} = \mathbf{0}$ which is possible only for $\partial_t \hat{\mathbf{v}} = \mathbf{0}$ on the interval. Therefore, for accelerated motion $\nabla \varphi_\infty$ is not identically zero on the interval and $\bar{\rho}_\infty$ is uniquely determined by $\hat{\mathbf{r}}, t_0$ and M_0 where M_0 is uniquely defined by (81) according to (72). ■

In the following Section 4 we present a class of examples where solutions of the KG equation concentrate at a trajectory which describes an accelerated motion.

4 Rectilinear accelerated motion of the wave with a fixed shape

We proved in the preceding section that if solutions of the KG equation concentrate at a trajectory, then the trajectory and the energy satisfy Newton's equation where the mass is defined by Einstein's formula. The results of the previous section are valid for general concentrating solutions of the KG equation. In this section we present a specific class of examples for which the concentration assumptions hold. The constructions are explicit and provide for examples of relativistic accelerated motion with the internal energy determined from the Klein-Gordon Lagrangian. To be able to construct an explicit example we make

the following simplifying assumptions: (i) the motion is rectilinear; (ii) the nonlinearity is logarithmic as in (20); (iii) the shape $|\psi|$ is fixed, namely it is Gaussian; (iv) the parameter $\zeta = \zeta_n$ in (61) satisfies condition $\zeta_n \rightarrow 0$. For a given rectilinear trajectory we find a sequence of parameters, potentials and solutions of the KG equation which concentrate at the trajectory. Now we formulate the assumptions in detail.

Recall that the parameters c, m, q are fixed. We assume that the sequence $a = a_n, \chi = \chi_n$ satisfies the conditions (60), (61) and an additional condition

$$\zeta = \zeta_n = \frac{a_c}{a} = \frac{\chi_n}{a_n m c} \rightarrow 0. \quad (123)$$

Trajectories $\hat{\mathbf{r}}(t)$ which describe *rectilinear* accelerated motion have the form

$$\hat{\mathbf{r}}(t) = (0, 0, r(t)), \quad -\infty < t < \infty. \quad (124)$$

We consider a fixed trajectory $r(t)$ such that corresponding velocity $v = \partial_t r$ is two times continuously differentiable and has uniformly bounded derivatives:

$$|v(t)| + |\partial_t v(t)| + |\partial_t^2 v(t)| \leq C, \quad -\infty < t < \infty. \quad (125)$$

We also impose a weaker version of the above restriction,

$$\sup_{\tau} (|\partial_{\tau} \beta| + |\partial_{\tau}^2 \beta|) \leq \hat{\epsilon}, \quad \text{where } \tau = \frac{c}{a} t, \quad (126)$$

β is the *normalized velocity*, namely

$$\beta = v/c, \quad v = \partial_t r, \quad (127)$$

this version is sufficient in many estimates. Since the parameter $a_n \rightarrow 0$, the assumption (126) is less restrictive than (125).

The velocity $v = \partial_t r$ is assumed to be smaller than the speed of light c , namely that

$$|\beta(t)| \leq \hat{\epsilon}_1 < 1, \quad -\infty < t < \infty; \quad (128)$$

we assume also that the normalized velocity does not vanish:

$$|\beta(t)| \geq \check{\beta} > 0, \quad -\infty < t < \infty. \quad (129)$$

Here is the main result of this section.

Theorem 4.1 *For any trajectory $\hat{\mathbf{r}}(t) = (0, 0, r(t))$ where r satisfies (125), (126), (128), (129), for any T_-, T_+ there exists a sequence a_n, R_n, χ_n , and potentials φ_n such that the KG equations are localized in $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$. There exists a sequence of solutions of the KG equation which concentrates at $\hat{\mathbf{r}}$.*

Proof. The statement follows from Theorem 4.16 which is proven at the end of this section. The potentials and solutions are described in the following sections. ■

4.1 Reduction to one dimension

When the external potential φ depends only on t and x_3 , the equation (3) in the three dimensional space with a logarithmic nonlinearity (21) can be reduced to a problem in one dimensional space by the following substitution

$$\psi = \pi^{-1/2} a^{-1} \exp \left(-a^{-2} (x_1^2 + x_2^2) / 2 \right) \psi_{1D}(t, x_3), \quad (130)$$

with ψ_{1D} being dependent only on x_3 and t . The corresponding reduced 1D KG equation for $\psi = \psi_{1D}$ with *one spatial variable* has the form

$$-c^{-2} \tilde{\partial}_t^2 \psi + \partial_3^2 \psi - G'_a(\psi^* \psi) \psi - \kappa_0^2 \psi = 0, \quad (131)$$

where

$$\kappa_0 = mc/\chi, \quad \tilde{\partial}_t = \partial_t + iq\varphi/\chi, \quad \varphi = \varphi(t, x_3),$$

and the 1D logarithmic nonlinearity takes the form

$$G'_a(|\psi|^2) = G'_{a,1D}(|\psi|^2) = -a^{-2} [\ln(\pi^{1/2} |\psi|^2) + 1] - a^{-2} \ln a. \quad (132)$$

From now on we write x instead of x_3 for the notational simplicity. We look for the external potential φ in the form

$$\varphi(t, x) = \varphi_{ac}(t, x) + \varphi_b(t, x; \zeta), \quad (133)$$

where the *accelerating potential* φ_{ac} is linear in y , namely

$$\varphi_{ac} = \varphi_0(t) + \varphi'_{ac} y, \quad y = x - r, \quad (134)$$

and $\varphi_b(t, x; \zeta)$ is a small *balancing potential*. The coefficient $\varphi'_{ac} = \partial_x \varphi_{ac}$ is determined by the trajectory $r(t)$ according to the formula

$$\partial_t(m\gamma v) + q\partial_x \varphi_{ac} = 0, \quad v = \partial_t r, \quad (135)$$

which has the form of *relativistic law of motion* (2). The potential φ_{ac} coincides in our construction with the limit potential φ_∞ in (65). The coefficient $\varphi_0(t)$ can be prescribed as an arbitrary function with bounded derivatives.

According to the above formula, φ_{ac} directly relates to the acceleration of the charge, and we call it "accelerating" potential, φ_{ac} does not depend on the small parameter ζ . The remaining part of the external potential φ in the KG equation (131) is a small "balancing" potential φ_b which allows the charge to exactly preserve its form as it accelerates. Below we find such a potential φ_b that the Gaussian wave function with the center $r(t)$ is a solution of (131) in the strip

$$\Xi(\theta_n) = \{(t, x) : |x - r(t)| \leq \theta_n a_n\}, \quad \theta_n \rightarrow \infty, \quad \theta_n a_n = R_n \rightarrow 0. \quad (136)$$

The balancing potential vanishes asymptotically, that is $\varphi_b(t, x; \zeta) \rightarrow 0$ as $\zeta \rightarrow 0$, and the forces it produces also become vanishingly small compared with the electric force $-q\partial_x \varphi_{ac}(x)$ in the strip $\Xi(\theta)$. Note that such an accelerated motion with a preserved shape is possible only for a properly chosen potential φ_b .

4.2 Equation in a moving frame

As the first step of the construction of the potential φ_b we rewrite the KG equation (131) in a moving frame. We take $\hat{\mathbf{r}}(t)$ as the new origin and make the following change of variables:

$$x_3 = r(t) + y, \quad \mathbf{x} = \hat{\mathbf{r}}(t) + \mathbf{y}. \quad (137)$$

The 1D KG equation (131) then takes the form

$$\begin{aligned} & -c^{-2} (\partial_t + iq\varphi/\chi - v\partial_y) (\partial_t + iq\varphi/\chi - v\partial_y) \psi \\ & + \partial_y^2 \psi - G'_a (\psi^* \psi) \psi - a_C^{-2} \psi = 0, \end{aligned} \quad (138)$$

The 1D logarithmic nonlinearity $G'_a = G'_{a,1D}$ is defined by (132), and the electric potential φ has the form (133). We assume that the solution $\psi(t, x)$ has the Gaussian form, namely

$$\begin{aligned} \psi &= \psi_{1D} = \hat{\Psi} \exp(i\hat{S}), \\ \hat{S} &= \omega_0 c^{-2} \gamma v y - s(t) - S(t, y), \quad \omega_0 = c/a_C = mc^2/\chi \end{aligned} \quad (139)$$

where we explicitly define the real valued function $\hat{\Psi}$:

$$\hat{\Psi} = a^{-1/2} \Psi(a^{-1}y) \quad (140)$$

where

$$\Psi(t, z) = \pi^{-1/4} e^{\sigma - z^2/2}, \quad (141)$$

with

$$\sigma = \sigma(t) = \ln \gamma^{-1/2}, \quad \gamma = (1 - \beta^2)^{-1/2}. \quad (142)$$

We define σ by the above formula to satisfy (153). Substitution of (139) into (138) yields

$$\begin{aligned} & -\frac{1}{c^2} \left(\partial_t + i\partial_t(\gamma v) \frac{\omega_0}{c^2} y - i\partial_t s + \frac{iq\varphi_0}{\chi} - iv^2 \frac{\omega_0}{c^2} \gamma \right. \\ & \quad \left. - i\partial_t S + iv\partial_y S + \frac{iq\varphi'_{ac}}{\chi} y + \frac{iq\varphi_b}{\chi} - v\partial_y \right)^2 \hat{\Psi} \\ & + \left(\partial_y - i\partial_y S + iv \frac{\omega_0}{c^2} \gamma \right)^2 \hat{\Psi} - G'_a (|\hat{\Psi}|^2) \hat{\Psi} - \kappa_0^2 \hat{\Psi} = 0. \end{aligned} \quad (143)$$

To eliminate the leading, independent of the small parameter ζ , terms in the above equation, we require (135) to hold together with the following equation:

$$-\partial_t s + q\varphi_0/\chi - v^2 \omega_0 c^{-2} \gamma = -\gamma \omega_0. \quad (144)$$

Observe that the expressions in (143) eliminated in view of equations (144) and equations (135) do not depend on ζ . Note that the constant part $\varphi_0(t)$ of the accelerating potential can be prescribed arbitrarily since we always can choose the phase shift $s(t)$ so that the equation (144) holds. Equation (138) combined with equations (144) and (135) can be transformed into

$$\begin{aligned} & -\frac{1}{c^2} \left(\partial_t - i\gamma \frac{c}{a_C} - i\partial_t S + iv\partial_y S + \frac{iq\varphi_b}{\chi} - v\partial_y \right)^2 \hat{\Psi} \\ & + \left(\partial_y - i\partial_y S + i \frac{1}{a_C} \frac{v}{c} \gamma \right)^2 \hat{\Psi} - G'_a (|\hat{\Psi}|^2) \hat{\Psi} - \frac{1}{a_C^2} \hat{\Psi} = 0. \end{aligned} \quad (145)$$

In the following sections, we find the quantities φ_b and S which are of order ζ^2 and satisfy this equation.

4.3 Equations for auxiliary phases

In this subsection we introduce two auxiliary phases and reduce the problem of determination of the potential φ_b and the phase S to a first-order partial differential equation for a single unknown phase. Solving this equation can be reduced to integration along characteristics allowing for a rather detailed mathematical analysis.

It is convenient to introduce rescaled dimensionless variables \mathbf{z}, τ :

$$\tau = \frac{c}{a}t, \quad \mathbf{z} = \frac{\zeta}{a_C}\mathbf{y} = \frac{1}{a}\mathbf{y}, \quad (146)$$

where we set $z_3 = z$. We introduce now auxiliary phases Z and Φ

$$Z = \zeta \partial_z S, \quad (147)$$

$$\Phi = -\zeta \partial_\tau S + \zeta \beta \partial_z S + \frac{qa_C \varphi_b}{c\chi}, \quad (148)$$

and these phases will be our new unknown variables. Obviously, if we find Z and Φ , we can find S by the integration in z and setting $S = 0$ at $z = 0$, see (226). After that φ_b can be found from (148). Consequently, to find a small φ_b we need to find small Z, Φ .

Equation (145) takes the following form:

$$\begin{aligned} & -(\zeta \partial_\tau + i\Phi - i\gamma - \beta \zeta \partial_z)^2 \Psi \\ & + (\zeta \partial_z - iZ + i\beta \gamma)^2 \Psi - \zeta^2 G'_1(\Psi^2) \Psi - \Psi = 0, \end{aligned} \quad (149)$$

where $\hat{\Psi}$ is given by (140), (141). We look for a solution of (149) in the strip $\Xi(\theta)$ in the time-space:

$$\Xi(\theta) = \{(\tau, z) : -\infty < \tau < \infty, \quad |z| < \theta\}. \quad (150)$$

We expand (149) with respect to Φ, Z and rewrite equation (149) in the form

$$\begin{aligned} & Q\Psi - i\Phi(\zeta \partial_\tau - i\gamma - \beta \zeta \partial_z) \Psi - i(\zeta \partial_\tau - i\gamma - \beta \zeta \partial_z)(\Phi\Psi) + \Phi^2\Psi \\ & + iZ(\zeta \partial_z + i\beta \gamma) \Psi + i(\zeta \partial_z + i\beta \gamma)(Z\Psi) - Z^2\Psi = 0, \end{aligned} \quad (151)$$

where we denote by $Q\Psi$ the term which does not involve Φ and Z explicitly:

$$Q = \frac{1}{\Psi} \left(-(\zeta \partial_\tau - i\gamma - \beta \zeta \partial_z)^2 \Psi + (\zeta \partial_z + i\beta \gamma)^2 \Psi \right) - \zeta^2 G'_1(\Psi^2) - 1. \quad (152)$$

Using (141) and (142) we conclude that the imaginary part of Q is zero:

$$\text{Im } Q = 2\zeta \gamma \partial_\tau \sigma + \zeta \partial_\tau \gamma = 0. \quad (153)$$

Hence, $Q = \text{Re } Q$ can be written in the form

$$Q = \zeta^2 \left(-\partial_\tau^2 \Psi + \partial_\tau \beta \partial_z \Psi + 2\beta \partial_\tau \partial_z \Psi + \gamma^{-2} \partial_z^2 \Psi - G'_1(\Psi^2) \right) \Psi / \Psi. \quad (154)$$

Now we show explicitly the dependence of Q on z . Using (132) we easily verify that the function Ψ in (141) satisfies an equation similar to (10):

$$\partial_z^2 \Psi - G'_a(\Psi \Psi^*) \Psi = 2\sigma \Psi. \quad (155)$$

Taking into account (155) we see that

$$Q = \zeta^2 \left[-(\partial_\tau^2 \sigma + (\partial_\tau \sigma)^2) - z \partial_\tau \beta - 2\beta z \partial_\tau \sigma - \beta^2 \zeta^2 (z^2 - 1) + 2\sigma \right]. \quad (156)$$

Now we rewrite the complex equation (151) as a system of two real equations. The real part of (151) divided by Ψ yields the following quadratic equation

$$Q - 2\gamma\Phi + \Phi^2 - 2\beta\gamma Z - Z^2 = 0. \quad (157)$$

The solution Z which is small for small Φ, Q is given by the formula

$$Z = \Theta(\Phi) = -\beta\gamma + (\Phi^2 - 2\gamma\Phi + \beta^2\gamma^2 + Q)^{1/2} \beta / |\beta|. \quad (158)$$

The imaginary part of (151) divided by $\zeta\Psi$ yields equation

$$-2\Phi(\partial_\tau - \beta\partial_z) \ln \Psi - (\partial_\tau - \beta\partial_z) \Phi + \partial_z Z + 2Z\partial_z \ln \Psi = 0 \quad (159)$$

where the coefficients are expressed in terms of Ψ defined by (140), (141):

$$\ln \Psi = \sigma - z^2/2, \quad \partial_z \ln \Psi = -z, \quad \partial_\tau \ln \Psi = \partial_\tau \sigma. \quad (160)$$

To determine a *small* solution Φ of (159), (158) in the strip Ξ we impose the condition

$$\Phi = 0 \text{ if } z = 0, \quad -\infty < \tau < \infty. \quad (161)$$

A solution Φ of the equation (159), where $Z = \Theta(\Phi)$ satisfies (158), is a solution of the following quasilinear first-order equation

$$\partial_\tau \Phi - \beta \partial_z \Phi - \Theta_\Phi(\Phi) \partial_z \Phi = -2\Phi \zeta (\partial_\tau - \beta \partial_z) \ln \Psi + 2\Theta \partial_z \ln \Psi + \Theta_z \quad (162)$$

where Θ_Φ and Θ_z are the partial derivatives of $\Theta(\Phi)$ (Θ depends on z via Q) with Φ subjected to condition (161).

To prove the existence of a solution in a wide enough strip Ξ and study its properties we use the method of characteristics.

4.4 Construction and properties of the auxiliary potential

We introduce the characteristic equations for (162):

$$dz/ds = -\Theta_\Phi(\Phi) - \beta(\tau), \quad (163)$$

$$d\tau/ds = 1, \quad (164)$$

$$d\Phi/ds = -2\Phi(\partial_\tau - \beta\partial_z) \ln \Psi + 2\Theta(\Phi) \partial_z \ln \Psi + \Theta_z, \quad (165)$$

with the initial data

$$\tau_{s=0} = \tau_0, \quad z_{s=0} = 0. \quad (166)$$

From (161) on the line $z = 0$ we derive the initial condition

$$\Phi_{s=0} = 0. \quad (167)$$

In the above equations the quantity Φ is an independent variable which coincides with $\Phi = \Phi(\tau, z)$ on the characteristic curves:

$$\Phi = \Phi(\tau(\tau_0, s), z(\tau_0, s)). \quad (168)$$

Equation (164) can be solved explicitly:

$$\tau = \tau_0 + s.$$

Often, abusing notation, we will write $\Phi(\tau_0, s)$ for the solution of (163)-(165) which is found independently from the formula (168). The function $\Theta(\Phi)$ and its partial derivatives Θ_z and Θ_Φ are determined by (158) and are well-defined and smooth as long as

$$D(\tau, z, \Phi) = \Phi^2 - 2\gamma(\tau)\Phi + \beta^2\gamma^2(\tau) + Q(\tau, z) > 0. \quad (169)$$

Then formula (158) determines the function Θ as an analytic function of Q, Φ . Taking into account (126) we see that the right-hand side of the system (163) (165) is a two time continuously differentiable function of variables τ, z, Φ in the domain in \mathbb{R}^3 defined by the inequality (169).

From the classical theorem on the existence, uniqueness and regular dependence on the initial data and parameters of solutions of the Cauchy problem for a system of ordinary differential equations (see, for instance, [15]), we readily obtain the following statement.

Lemma 4.2 *Let the initial data*

$$\tau_{s=0} = \tau_0, \quad z_{s=0} = z_0, \quad \Phi_{s=0} = \Phi_0 \quad (170)$$

be such that the function D defined by (169) is positive, namely $D(\tau_0, z_0, \Phi_0) > 0$. Then the system (163), (165) with the initial condition (170) has a unique solution $\tau(s), z(s), \Phi(s)$ defined on a maximal (finite or infinite) interval $s_- < s < s_+$. If the value s_\pm is finite, then

$$\text{either } \lim_{s \rightarrow s_\pm} (|\Phi|^2 + z^2) = \infty \text{ or } \lim_{s \rightarrow s_\pm} D(\tau, z, \Phi) = 0. \quad (171)$$

The solution is a twice continuously differentiable function of s , of the initial data τ_0, z_0, Φ_0 , and of the parameter ζ , and $\tau(s) = \tau_0 + s$.

4.4.1 Properties of the characteristic curves

The characteristic system (163)–(165), (167) involves the function $Z = \Theta(\Phi)$ defined by (158). We give sufficient conditions on the variables which ensure that (169) holds.

Proposition 4.3 *If*

$$|\Phi| \leq \check{\beta}^2/4 \quad (172)$$

and

$$Q \geq -\check{\beta}^2/4 \quad (173)$$

then

$$D(\tau, z, \Phi) \geq \Phi^2 + \check{\beta}^2/4, \quad (174)$$

and (169) holds. Conditions (172) and (173) are fulfilled if

$$\zeta^{1/3} |z| \leq 1, \quad (175)$$

$$\zeta^2 \sup_{\tau} |\partial_\tau^2 \sigma + (\partial_\tau \sigma)^2 - 2\sigma - \beta^2 \zeta^2| + \zeta^{4/3} \sup_{\tau} |\partial_\tau \beta + 2\beta \partial_\tau \sigma| + \zeta^{10/3} \leq \check{\beta}^2/8. \quad (176)$$

Proof. If (172) and (173) hold, we take into account that $\gamma \geq 1$ and conclude that

$$\Phi^2 - 2\gamma\Phi + Q + \beta^2\gamma^2 \geq \Phi^2 + \check{\beta}^2/4 > 0. \quad (177)$$

According to (156) condition (173) takes the form

$$\zeta^2 (\partial_\tau^2 \sigma + (\partial_\tau \sigma)^2 - 2\sigma - \beta^2 \zeta^2) + z\zeta^2 (\partial_\tau \beta + 2\beta \partial_\tau \sigma) + \zeta^4 z^2 \beta^2 \leq \check{\beta}^2/4.$$

Hence (173) is fulfilled if (175), (176) hold. ■

Therefore $\Theta(\Phi)$ is a regular function of Φ which satisfies inequality (172) in the strip $\Xi(\theta)$ defined by (150) as long as ζ satisfies (176).

It is convenient to introduce the following notation. Let 2_+ be a fixed number which is arbitrary close to 2 and 1_+ be arbitrarily close to 1 and satisfy the inequality

$$1 < 2_+/2 < 1_+ < 2. \quad (178)$$

Below we obtain estimates of solutions of the characteristic equations in the strip $\Xi(\bar{\theta})$ using notation

$$\bar{\theta} = \bar{\theta}(\zeta) = \ln^{1/2_+} \zeta^{-1}. \quad (179)$$

It is also convenient to introduce the following functions:

$$b(\tau) = \beta^{-1}(\tau) - \beta(\tau), \quad B(\tau_0, s) = \int_{\tau_0}^{\tau_0+s} b(\tau) d\tau. \quad (180)$$

Using the inequality

$$0 < \hat{\epsilon}_1^{-1} - \hat{\epsilon}_1 \leq |\beta^{-1} - \beta| = (\beta^{-1} - \beta) \beta / |\beta| \leq \check{\beta}^{-1} - \check{\beta} \quad (181)$$

we obtain that

$$(\hat{\epsilon}_1^{-1} - \hat{\epsilon}_1) |s| \leq B(\tau_0, s) \leq |s| (\check{\beta}^{-1} - \check{\beta}). \quad (182)$$

Lemma 4.4 *Let $\zeta \leq 1/C_0$ where C_0 is sufficiently large. Let τ, z, Φ be a solution to (163)-(167) and assume that $z(s), \Phi(s)$ are defined on an interval $s_{1-} < s < s_{1+}$ with values in the strip $\Xi(\bar{\theta})$, and that the following estimate holds*

$$|\Phi(\tau(s), z(s))| \leq |\zeta| \quad \text{for } s_{1-} < s < s_{1+}. \quad (183)$$

Then there exist a constant $C_5 > 0$ such that on the same interval we have:

$$0 < (\hat{\epsilon}_1^{-1} - \hat{\epsilon}_1) / 2 \leq |dz/ds| \leq 2 (\check{\beta}^{-1} - \check{\beta}), \quad (184)$$

$$(\hat{\epsilon}_1^{-1} - \hat{\epsilon}_1) |s| / 2 \leq |z| \leq 2 |s| (\check{\beta}^{-1} - \check{\beta}), \quad (185)$$

$$|z - B(\tau_0, s)| \leq C\zeta |s|, \quad (186)$$

$$|\Phi| \leq C_5 \zeta^2 (|z|^2 + |z|) e^{|z|^2}. \quad (187)$$

Proof. If (183) holds, conditions (175), (176) are satisfied for $\zeta \leq 1/C_0$, implying that (173) and (174) are fulfilled. Equation (163) can be written in the form

$$dz/ds = \beta^{-1} - \beta - \Theta_{\Phi}^1(\Phi) \quad (188)$$

where

$$\begin{aligned} \Theta_{\Phi}^1 &= \Theta_{\Phi}(\Phi) + \beta^{-1} \\ &= \frac{\Phi}{((\Phi - \gamma)^2 - 1 + Q)^{1/2}} \frac{\beta}{|\beta|} + \frac{\Phi^2 - 2\gamma\Phi + Q}{\left(((\Phi - \gamma)^2 - 1 + Q)^{1/2} + \gamma|\beta| \right) ((\Phi - \gamma)^2 - 1 + Q)^{1/2}}. \end{aligned} \quad (189)$$

Evidently Θ_{Φ}^1 has the form

$$\Theta_{\Phi}^1 = \Phi U_1 + Q U_2, \quad (190)$$

where U_1 and U_2 are algebraic expressions which are analytic if (174) is fulfilled, and consequently they are bounded. Using (156) with $|z| \leq \bar{\theta}$ and small ζ we conclude that

$$|Q| = \zeta^2 |\partial_{\tau}^2 \sigma + (\partial_{\tau} \sigma)^2 - 2\sigma - \beta^2 \zeta^2 + z(\partial_{\tau} \beta + 2\beta \partial_{\tau} \sigma) + \zeta^2 z^2 \beta^2| \leq C_0 \zeta^2 (|z| + 1), \quad (191)$$

and obtain an estimate

$$|\Theta_{\Phi}^1| \leq C_1 \zeta \text{ for } s_{1-} < s < s_{1+}. \quad (192)$$

Hence, (163) implies that

$$|dz/ds - \beta^{-1} + \beta| \leq C \zeta^{2-\delta}. \quad (193)$$

We take ζ so small that $2C\zeta \leq \min(\hat{\epsilon}_1^{-1} - \hat{\epsilon}_1, \check{\beta}^{-1} - \check{\beta})$, and using (181) we obtain (184). Then integration implies (185) and (186).

Observe that equation (165) can be written in the form

$$\frac{d\Phi}{ds} = -2\Phi \partial_{\tau} \ln \Psi + 2(\Theta(\Phi) - \Phi \Theta_{\Phi}(\Phi)) \partial_z \ln \Psi + 2\Phi(\Theta_{\Phi}(\Phi) + \beta) \partial_z \ln \Psi + \Theta_z,$$

which, in turn, with the use of (163) can be rewritten as

$$\frac{d\Phi}{ds} = -2\Phi \partial_{\tau} \ln \Psi + 2(\Theta(\Phi) - \Phi \Theta_{\Phi}(\Phi)) \partial_z \ln \Psi - 2\Phi \frac{dz}{ds} \partial_z \ln \Psi + \Theta_z. \quad (194)$$

According to (173) $\beta^2 \gamma^2 + Q > 0$, and we rewrite (165) in the form

$$\frac{d\Phi}{ds} = -\Phi \frac{d \ln \Psi^2}{ds} + 2\Theta^1(\Phi) \partial_z \ln \Psi^2 - 2\left(\beta\gamma - (\beta^2 \gamma^2 + Q)^{1/2} \beta/|\beta|\right) \partial_z \ln \Psi^2 + \Theta_z \quad (195)$$

where

$$\Theta^1(\Phi) = \Theta(\Phi) - \Phi \Theta_{\Phi}(\Phi) + \beta\gamma - (\beta^2 \gamma^2 + Q)^{1/2} \beta/|\beta|. \quad (196)$$

Multiplying by Ψ^2 we obtain

$$\frac{d(\Psi^2 \Phi)}{ds} = 2\Theta_1(\Phi) z - 2\left(\beta\gamma - (\beta^2 \gamma^2 + Q)^{1/2}\right) z + \Theta_z \Psi^2. \quad (197)$$

According to (158)

$$\begin{aligned} \frac{\beta}{|\beta|} \Theta^1(\Phi) = & - \frac{\Phi^2}{((\Phi - \gamma)^2 - 1 + Q)^{1/2}} \frac{(\beta^2 \gamma^2 + Q)^{1/2}}{(\beta^2 \gamma^2 + Q)^{1/2} + ((\Phi - \gamma)^2 - 1 + Q)^{1/2}} \\ & - \gamma \Phi^2 \frac{\Phi - 2\gamma}{\left[(\beta^2 \gamma^2 + Q)^{1/2} + ((\Phi - \gamma)^2 - 1 + Q)^{1/2} \right]^2 ((\Phi - \gamma)^2 - 1 + Q)^{1/2}}. \end{aligned} \quad (198)$$

Hence, taking into account (174) we obtain

$$|\Theta^1(\Phi)| \leq C_1 \Phi^2.$$

To estimate remaining terms in (197) we note that in $\Xi(\bar{\theta})$

$$\left| \beta \gamma - (\beta^2 \gamma^2 + Q)^{1/2} \right| \leq C |Q| \leq C' \zeta^2 |z|. \quad (199)$$

Using (174) we obtain in $\Xi(\bar{\theta})$ an estimate

$$|\Theta_z(\Phi)| = \frac{|\zeta^2 (\partial_\tau \beta + 2\beta \partial_\tau \sigma) + 2\zeta^4 z \beta^2|}{2 (\Phi^2 - 2\gamma \Phi + \beta^2 \gamma^2 + Q)^{1/2}} \leq C_1 \zeta^2. \quad (200)$$

The above estimates combined with (197) and (200) yield

$$\left| \frac{d(\Psi^2 \Phi)}{ds} \right| \leq C_2 \Phi^2 |z| + C'' \zeta^2 |z| + C_3 \zeta^2. \quad (201)$$

Using (185) we obtain

$$|\Psi^2 \Phi| \leq C_4 \zeta^2 s^2 + C_3 \zeta^2 |s|,$$

and, using (185) and the definition of Ψ , we obtain (187). ■

Theorem 4.5 *Let δ be an arbitrary small fixed number satisfying $0 < \delta < \frac{2+}{32}$, and suppose that $\zeta \leq 1/C_0$ is sufficiently small. Let $s_{2-} < s < s_{2+}$ be a maximal interval on which a solution τ, z, Φ of (163)- (167) is such that $z(s), \tau(s)$ takes values in the strip $\Xi(\bar{\theta})$ where $\bar{\theta} = \ln^{1/2+} \zeta^{-1}$ and $|\Phi(s)| \leq \zeta$. Then*

$$z(s_{2+}) = \bar{\theta} \beta / |\beta|, \quad z(s_{2-}) = -\bar{\theta} \beta / |\beta|, \quad \bar{\theta} = \ln^{1/2+} \zeta^{-1}, \quad (202)$$

and $z(s), \Phi(s)$ satisfy inequalities (174), (185) on this interval as well as the inequality

$$|\Phi| \leq \zeta^{2-\delta}, \quad \left| \frac{d\Phi}{ds} \right| \leq C_7 \zeta^{2-\delta}. \quad (203)$$

The constants C_7, C_0 depend on δ but do not depend on τ_0, ζ .

Proof. Consider an interval (s_{1-}, s_{1+}) satisfying the conditions of Lemma 4.4. According to Lemma 4.2 such an interval exists. Consider the maximal interval (s_{1-}, s_{1+}) such that $z(s), \Phi(s)$ takes values in the strip $\Xi(\bar{\theta})$ and conditions of Lemma 4.4 are fulfilled, and denote the maximal value of s_{1+} by s_{2+} and the minimal value of s_{1-} by s_{2-} . According to Lemma 4.4 inequality (174) is satisfied on every (s_{1-}, s_{1+}) , and hence it is satisfied on

(s_{2-}, s_{2+}) . On such an interval (187) is fulfilled, and taking into account that $|z| \leq \ln^{1/2+} \zeta^{-1}$ we obtain (203). Note that (187) implies that

$$|\Phi| \leq 2C_5 \zeta^2 \ln^{2/2+} \zeta^{-1} \exp \ln^{2/2+} \zeta^{-1} \leq \zeta^{2-\delta}/2 \quad (204)$$

if ζ is small enough to satisfy

$$C_5 \ln^{2/2+} \zeta^{-1} \exp \left(\ln^{2/2+} \zeta^{-1} - \delta \ln \zeta^{-1} \right) \leq 1/2.$$

Hence (183) holds under the above condition on ζ which evidently does not depend on τ_0 . We assert that (202) is fulfilled. Indeed, assume the contrary. If

$$|z(s_{2+})| < \ln^{1/2+} \zeta^{-1},$$

we can extend the solution to a small interval with $|s| > |s_{2+}|$. The solution stays in $\Xi(\bar{\theta})$ by continuity and (204) implies (183) by continuity for small $s - s_{2+}$. This contradicts the assumption that s_{2+} is maximal. The quantity s_{2-} can be treated similarly yielding (202). The first inequality (203) follows from (204). To obtain the inequality for the derivative we use (201) and (204) as follows:

$$|d\Phi/ds| \leq C_5 s \zeta^{4-2\delta} + C_3 \zeta^2 + C_6 |\Phi| \leq C_7 \zeta^{2-\delta}.$$

■

We denote by $s_{2\pm}(\tau_0)$ the value of $s_{2\pm}$ given by (202) which was found in Theorem 4.5. According to (184) and the implicit function theorem the function $s_{2\pm}(\tau_0)$ is differentiable. Let us introduce a set

$$\Xi'(\bar{\theta}) = \{(\tau_0, s) \in \mathbb{R}^2 : \tau_0 \in \mathbb{R}, \quad s_{2-} \leq s \leq s_{2+}\}, \quad (205)$$

where according to (185)

$$C^{-1} \ln^{1/2+} \zeta^{-1} \leq |s_{2\pm}| \leq C \ln^{1/2+} \zeta^{-1}. \quad (206)$$

4.4.2 Properties of the auxiliary potentials

Based on the properties of solutions to the characteristic equations we establish here properties of solutions to the quasilinear equation (162). Notice that at $\zeta = 0$ the characteristic equations are linear and the case of small ζ can be considered in $\Xi(\bar{\theta})$ as a small perturbation.

We make use below of C^l norms of a function of two variables defined as follows:

$$\begin{aligned} \|u\|_{C^l(\Omega)} &= \sup_{y \in \Omega} \sum_{|\alpha| \leq l} |\partial^\alpha u(y)|, \text{ where } y = (y_1, y_2), \\ \partial^\alpha &= \partial_1^{\alpha_1} \partial_2^{\alpha_2}, \alpha = (\alpha_1, \alpha_2), |\alpha| = \alpha_1 + \alpha_2. \end{aligned} \quad (207)$$

If $\Omega = \Xi'(\bar{\theta})$ we set in the formula above $y = (\tau_0, s)$, whereas if $\Omega = \Xi(\bar{\theta})$ we set $y = (\tau, z)$. Let $B(\tau_0, s)$ be defined by (180). Obviously,

$$|\partial^\alpha B(\tau_0, s)| \leq C, \quad 1 \leq |\alpha| \leq 2. \quad (208)$$

Lemma 4.6 *Under conditions of Theorem 4.5 functions $z(\tau_0, s; \zeta)$ and $\Phi(\tau_0, s; \zeta)$ satisfy in $\Xi'(\bar{\theta})$, $\bar{\theta} = \ln^{1/2} \zeta^{-1/2+}$, estimates*

$$\|z - B\|_{C^l(\Xi'(\bar{\theta}))} \leq C\zeta^{2-\delta l-\delta}, \quad l = 0, 1, 2; \quad (209)$$

$$\|\Phi(\tau_0, s)\|_{C^l(\Xi'(\bar{\theta}))} \leq C\zeta^{2-\delta l-\delta}, \quad l = 0, 1, 2. \quad (210)$$

Proof. The derivation of the above estimates is straightforward but tedious, and we present only the principal steps. The estimates are derived by induction in l . For $l = 0$ we use Theorem 4.5, and then inequality (210) follows from (203). According to (203) inequality (183) is fulfilled and then inequality (209) for $l = 0$ follows from (186) and (185). Consider now $l > 0$ assuming that (209), (210) hold for $l - 1$. Equations (188), (195) can be written in the form

$$\frac{d(z - B)}{ds} = -\Phi U_{11} - Q U_{12}, \quad (211)$$

$$\frac{d\Phi}{ds} = -\Phi(\partial_\tau \sigma - z U_{20}) + \Phi^2 U_{21} z + Q U_{22} z + \zeta^2 U_{23} z + \zeta^2 U_{24}, \quad (212)$$

where U_{ij} are algebraic functions of variables Φ, Q, γ, β . These variables are bounded, and the derivatives of γ, β, σ are bounded as well by (126). The derivatives of the solutions, namely $z' = \partial^\alpha z, \Phi' = \partial^\alpha \Phi, |\alpha| = l$, satisfy the equations obtained by application of ∂^α to (188), (195). Since (174) is fulfilled, the coefficients U_{ij} and their derivatives with respect to Φ, Q, γ, β are bounded. The only unbounded variables are z and s according to (185), and their upper bounds are respectively $\bar{\theta}$ and $C\bar{\theta}$. Observe that z enters U_{11} and U_{12}, U_{21} only through Q . The derivatives of U_{ij} or of Q up to l -th order cannot involve powers of z higher than z^{2l} . Since Q given by (156) involves the factor ζ^2 and $|z| \leq \bar{\theta}$ in $\Xi'(\bar{\theta})$, the derivatives of Q of order l with respect to z or τ are smaller than $\zeta^{2-\delta}$ for small ζ , and derivatives of U_{ij} with respect to z or τ are smaller than $\zeta^{2-l\delta}$. We apply the Leibnitz formula to the derivatives of (211) and (212). Using the induction assumption

$$\|\Phi\|_{C^{l-1}(\Xi'(\bar{\theta}))} \leq C\zeta^{2-l\delta}, \quad \|z - B\|_{C^{l-1}(\Xi'(\bar{\theta}))} \leq C\zeta^{2-l\delta},$$

and notation (180), we can write the equations in the form

$$\frac{dz'}{ds} = \partial^\alpha b + U'_{11}\Phi' + U'_{12}z' + U_{13,\alpha}, \quad (213)$$

$$\frac{d\Phi'}{ds} = U'_{21}\Phi' + U'_{22}z' + U_{23,\alpha}, \quad (214)$$

where

$$\begin{aligned} |U'_{11}| &\leq C_1, & U'_{12} &\leq \zeta^{2-\delta}, & U_{13,\alpha} &\leq \zeta^{2-l\delta}, \\ |U'_{21}| &\leq C_0\bar{\theta}, & |U'_{22}| &\leq C\zeta^{2-l\delta}, & U_{23,\alpha} &\leq C\zeta^{2-l\delta}. \end{aligned} \quad (215)$$

The initial data for $z' = \partial^\alpha z, \Phi' = \partial^\alpha \Phi$ can also be analyzed by induction, since $\partial_{\tau_0}^i \Phi_{s=0} = 0$, $\partial_{\tau_0}^i z_{s=0} = 0$ and $\partial_s^j \partial_{\tau_0}^i \Phi$ can be expressed from the equation for $\partial_s^{j-1} \partial_{\tau_0}^i \Phi$. Note that $\partial^\alpha b = \partial_\tau^{|\alpha|} b = \partial_s^{|\alpha|} b$ and (213) can be rewritten in the form

$$\frac{d}{ds} (z' - \partial_s^{|\alpha|-1} b) = U'_{11}\Phi' + U'_{12} (z' - \partial_s^{|\alpha|-1} b) + U'_{12} \partial_s^{|\alpha|-1} b + U_{13,\alpha}. \quad (216)$$

Hence, we get from the induction assumptions estimates of the initial data for small ζ

$$|\partial^\alpha z_{s=0} - \partial_s^{|\alpha|-1} b_{s=0}| \leq C_1 \zeta^{2-l\delta}, \quad |\partial^\alpha \Phi_{s=0}| \leq C_2 \zeta^{2-l\delta}, \quad |\alpha| = l.$$

From the system (216), (214) we easily obtain for small ζ the following estimate

$$|z' - \partial_s^{|\alpha|-1} b| + |\Phi'| \leq C_3 \zeta^{2-l\delta} \exp(C'_0 \bar{\theta} s) \leq C_3 \zeta^{2-l\delta} \exp\left(C'_0 \ln^{2/2+} \zeta^{-1}\right) \leq \zeta^{2-l\delta-\delta} \quad (217)$$

which implies desired (209) and (210). ■

Notice that solutions of the characteristic equations determine the function $\Phi(\tau(\tau_0, s), z(\tau_0, s))$ on the characteristic curves as a function of parameters τ_0, s . Characteristic equations (163)-(167) also determine a function $\Pi(\tau_0, s) = (\tau(\tau_0, s), z(\tau_0, s))$ in (τ, z) -plane. To obtain the function Φ of independent variables (τ, z) we have to find the inverse of Π , that is $(\tau_0, s) = \Pi^{-1}(\tau, z)$ in a strip about the line $z = 0$.

Lemma 4.7 *The image of the mapping Π contains the strip $\Xi(\bar{\theta})$ if $\zeta \leq 1/C$ for sufficiently large constant C .*

Proof. The mapping Π maps the straight line $\{\tau_0, s = 0\}$ onto the straight line $\{\tau, z = 0\}$. The straight line $\{\tau_0 = \tau_{00}, s\}$ is mapped onto the curve $\tau = \tau_{00} + s, z = z(\tau_{00}, s)$. This curve intersects straight line $|z_0| = \bar{\theta}$ and, consequently, any straight line $z = z_0$ with $|z_0| \leq \bar{\theta}$. This intersection is transversal according to (184). Hence the point of intersection $\tau = p(\tau_{00}, z_0)$ continuously and differentially depends on τ_{00} . Formula $\tau = \tau_{00} + s$ implies that $p(\tau_{00}, z_0) \rightarrow \pm\infty$ as $\tau_{00} \rightarrow \pm\infty$. Since $p(\tau_{00}, z_0)$ is a continuous function, it takes all intermediate values on the straight line, therefore the image of the mapping Π contains every straight line $z = z_0$ with $|z_0| \leq \bar{\theta}$. ■

Now we want to prove that the mapping Π is one-to-one on $\Xi'(\bar{\theta})$ and that its inverse has uniformly bounded derivatives in the strip $\Xi(\bar{\theta})$. The characteristic system depends on the small parameter ζ . Therefore $\Pi(\tau_0, s) = \Pi(\tau_0, s; \zeta)$ and its differential is given by the matrix

$$\Pi'(\tau_0, s; \zeta) = \begin{pmatrix} \partial\tau/\partial\tau_0 & \partial\tau/\partial s \\ \partial z/\partial\tau_0 & \partial z/\partial s \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \partial z/\partial\tau_0 & \partial z/\partial s \end{pmatrix}. \quad (218)$$

If the matrix determinant is not zero, the inverse is given by the formula

$$\Pi'^{-1}(\tau_0, s; \zeta) = \frac{1}{\partial z/\partial s - \partial z/\partial\tau_0} \begin{pmatrix} \partial z/\partial s & -1 \\ -\partial z/\partial\tau_0 & 1 \end{pmatrix}.$$

For $\zeta = 0$ we obtain from (163)-(165) a simpler system for the resulting approximation $\mathring{\Phi}$:

$$\frac{d\tau}{ds} = 1, \quad dz/ds = (\beta^{-1} - \beta)(\tau_0 + s), \quad (219)$$

$$\frac{d\mathring{\Phi}}{ds} = -2\mathring{\Phi}(\partial_\tau - \beta\partial_z) \ln \Psi - 2\beta^{-1}\mathring{\Phi}\partial_z \ln \Psi. \quad (220)$$

The solution of (219) is given by the formula

$$\tau = \tau_0 + s, \quad z = z_0 = B(\tau_0, s), \quad (221)$$

where $B(\tau_0, s), b(\tau)$ are given by (180). The differential of $\Pi(\tau_0, s; 0)$ is given by the matrix

$$\Pi'(\tau_0, s; 0) = \begin{pmatrix} 1 & 1 \\ b(\tau_0 + s) - b(\tau_0) & b(\tau_0 + s) \end{pmatrix} \quad (222)$$

with the determinant $\det \Pi(\tau_0, s; 0) = b(\tau_0)$. The matrix is invertible according to (181), and the inverse matrix is uniformly bounded.

When $\zeta > 0$ is small, we consider the system (163)-(165), (167) as a perturbation of the system with $\zeta = 0$ and the differential $\Pi'(\tau_0, s; \zeta)$ is also a small perturbation of $\Pi'(\tau_0, s; 0)$.

Lemma 4.8 *Let conditions of Theorem 4.5 be satisfied. Then in $\Xi(\bar{\theta})$*

$$|\Pi'(\tau_0, s; \zeta) - \Pi'(\tau_0, s; 0)| \leq C\zeta^{2-\delta} \quad (223)$$

for $|\zeta| \leq 1/C_0$, the matrices $\Pi'(\tau_0, s; \zeta)$ are invertible, the inverse matrices $\Pi'^{-1}(\tau_0, s; \zeta)$ have continuously differentiable elements and their derivatives are uniformly bounded in $\Xi(\bar{\theta})$. The mapping Π is one-to-one from $\Xi'(\bar{\theta})$ to $\Xi(\bar{\theta})$, the mappings Π and Π^{-1} are two times differentiable with uniformly bounded derivatives.

Proof. We use Lemma 4.6 and infer from (209) with $l = 1$ that

$$|\partial z / \partial \tau_0 - \partial z_0 / \partial \tau_0| + |\partial z / \partial s - \partial z_0 / \partial s| \leq C\zeta^{2-\delta}.$$

This inequality implies inequality (223). Inequality (223), in turn, implies that $\Pi'(\zeta) = \Pi'(\tau_0, s; \zeta)$ is a uniformly small perturbation of the invertible matrix (222) $\Pi'(0)$, and hence $\Pi'^{-1}(\zeta)$ is a uniformly small perturbation of the matrix $\Pi'^{-1}(0)$. The matrices $\Pi'(\zeta)$ and $\Pi'^{-1}(\zeta)$ have continuously differentiable entries. Since $|\Pi'^{-1}(0)|$ is uniformly bounded and the derivatives of entries of $\Pi'(0)$ are uniformly bounded and continuous in $\Xi(\bar{\theta})$, then the derivatives of entries of $\Pi'^{-1}(0)$ are uniformly bounded and continuous in $\Xi(\bar{\theta})$ as well. The mapping $\Pi(\zeta)$ is one-to-one since it is a local diffeomorphism between $\Xi'(\bar{\theta})$ and $\Xi(\bar{\theta})$ and the image is a simply connected domain. Since Π is two times continuously differentiable with uniformly bounded derivatives and $\Pi'^{-1}(\zeta)$ is uniformly bounded, the inverse mapping $\Pi^{-1}(\zeta)$ is two times continuously differentiable with uniformly bounded derivatives. ■

Theorem 4.9 *For any $\delta > 0$ there exists such C_0 that if $\zeta \leq 1/C_0$ then there exists a solution $\Phi(\tau, z)$ of the quasilinear equation (162) defined in $\Xi(\bar{\theta})$. This solution is twice continuously differentiable in the strip $\Xi(\bar{\theta})$ and its derivatives are uniformly bounded and small for $\zeta \leq 1/C_0$, namely*

$$\|\Phi(\tau, z)\|_{C^2(\Xi(\bar{\theta}))} \leq C_1 \zeta^{2-3\delta}. \quad (224)$$

Proof. The solution $\Phi(\tau, z)$ is defined by formula (168) which can be written in the form

$$\Phi(\tau, z; \zeta) = \Phi(\Pi^{-1}(\tau_0, s; \zeta), \zeta). \quad (225)$$

The function $\Phi(\tau, z; \zeta)$ is well-defined in $\Xi(\bar{\theta})$ according to Lemmas 4.7 and 4.8. Its differentiability properties follow from properties of $\Phi(\tau_0, s; \zeta)$ described in (210) and properties of $\Pi^{-1}(\tau_0, s; \zeta)$ described in Lemma 4.8. It is a solution to (162) according to the construction of $\Phi(\tau_0, s; \zeta)$ as a solution of (163)-(165). ■

The second auxiliary phase Z is given by the formula (158). Since (174) holds in $\Xi(\bar{\theta})$, $Z(\tau, z)$ is also twice continuously differentiable in the strip $\Xi(\bar{\theta})$ and has uniformly bounded derivatives. The potential φ_b and phase S can be found from (147), (148), namely

$$S = \zeta^{-1} \int_0^z Z(\tau, z_1) dz_1 = \zeta^{-1} \int_0^z \Theta(\Phi) dz_1, \quad (226)$$

and

$$\varphi_b = \frac{mc^2}{q} \left(\Phi + \partial_\tau \int_0^z \Theta(\Phi) dz_1 - \beta \Theta(\Phi) \right). \quad (227)$$

Lemma 4.10 *The phase function S defined by (226) and the potential φ_b defined by (227) satisfy the estimates*

$$\|S\|_{C^2(\Xi(\bar{\theta}))} \leq C_1 \zeta^{1-4\delta}, \quad (228)$$

$$\|\varphi_b\|_{C^1(\Xi(\bar{\theta}))} \leq C_2 \zeta^{2-4\delta}. \quad (229)$$

Proof. The above estimates follow from (224) and the boundedness in $\Xi'(\bar{\theta})$ of derivatives of the function $\Theta(\Phi)$ which enters representations (226) and (227). ■

4.5 Verification of the concentration conditions

In this section we fix an interval $T_- \leq t \leq T_+$ and define sequences $a = a_n$, $\zeta = \zeta_n$, $R = R_n$. We verify then that the KG equation is localized at the trajectory $\hat{\mathbf{r}}(t) = (0, 0, r(t))$ and that solutions ψ defined by (130), (139) concentrate at $\hat{\mathbf{r}}(t)$ as in Theorem 4.16).

Proposition 4.11 *Let $\varphi(t)$ be defined by (133)-(135) where φ_b is given by (227) and $\varphi_0(t)$ satisfy (243), $\varphi_\infty = \varphi_{ac}(t)$ given by (134), (135), $R_n = \bar{\theta}a_n \rightarrow 0$. Let also*

$$a_C \leq Ca^2. \quad (230)$$

Then φ_n satisfy (63) and (64), and condition (66) holds.

Proof. Note that $\varphi(t, x)$ does not depend on x_1, x_2 and estimates in $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$ follow from estimates in $\Xi(\bar{\theta})$:

$$\max_{\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)} |\varphi(t, x)| \leq \sup_{\tau, z \in \Xi(\bar{\theta})} |\varphi(\tau, z)|, \quad (231)$$

$$\max_{\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)} |\partial_t \varphi(t, x)| \leq \frac{c}{a} \sup_{\tau, z \in \Xi(\bar{\theta})} |\partial_\tau \varphi|, \quad \max_{\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)} |\nabla \varphi(t, x)| \leq \frac{1}{a} \sup_{\tau, z \in \Xi(\bar{\theta})} |\nabla_z \varphi|.$$

According to (133)

$$|\varphi(t, x)| \leq |\varphi_0| + |\varphi'_{ac}| |y| + |\varphi_b| \leq |\varphi_0| + |\varphi'_{ac}| R_n + |\varphi_b|.$$

Since φ'_{ac} is defined by (135), we conclude using (229), (243) that $|\varphi| \leq C_1$ in $\Omega(\hat{\mathbf{r}}(t), R_n)$. Using (230) we obtain estimate of derivatives

$$|\partial_t \varphi(t, x)| \leq |\partial_t \varphi_0| + |\partial_t \varphi'_{ac}| |y| + ca^{-1} |\partial_\tau \varphi_b| \leq C + C_1 a^{-1} \zeta^{2-4\delta} \leq C + C_2 a^{1-4\delta} \leq C_3, \quad (232)$$

$$|\nabla_x \varphi(t, x)| \leq |\varphi'_{ac}| + a^{-1} |\nabla_z \varphi_b| \leq C' + C'_1 a^{-1} \zeta^{2-4\delta} \leq C'_3 \quad (233)$$

Hence, (63) holds. According to (65), (133) and (134) $\varphi - \varphi_\infty = \varphi_b$. Using (229), (231), and observing that (230) implies $\zeta \leq Ca$, we conclude that

$$|\varphi_b(t, x)| + |\nabla_{0,x} \varphi_b(t, x)| \leq \sup_{\tau, z \in \Xi(\bar{\theta})} (|\varphi_b(\tau, z)| + \frac{1}{a} |\partial_\tau \varphi_b| + \frac{1}{a} |\nabla_z \varphi_b|) \leq C_4 a^{-1} \zeta^{2-4\delta} \rightarrow 0$$

yielding relations (64). To obtain (66) we use (125) and (135). ■

The solution of the KG equation is given by the formula (139), (130) where $s(t)$ is given by (144) with condition $s(0) = 0$.

Proposition 4.12 *Let ψ be defined by (130), (139) where s, S are defined by (144), (135), let $R_n = \bar{\theta} a_n$. Then in $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$*

$$|\partial_t \psi|^2 + |\nabla \psi|^2 \leq C_1 a_C^{-2} \dot{\psi}^2, \quad (234)$$

$$G(|\psi|^2) \leq C_2 \zeta^{2-\delta} a_C^{-2} \dot{\psi}^2, \quad (235)$$

where

$$\dot{\psi} = \dot{\psi}(\mathbf{y}/a) = \dot{\psi}(\mathbf{z}) = \pi^{-3/4} a^{-3/2} e^{-|\mathbf{y}|^2 a^{-2}/2}.$$

Proof. According to (130), (139) $\psi = e^{i\hat{S}(t, (x-r))} \hat{\Psi}$ with $\hat{\Psi} = e^\sigma \dot{\psi}(\mathbf{x} - \hat{\mathbf{r}})$. For such solutions we use the change of variables (137) and relations (135) and (144), and we obtain similarly to (143) and (145)

$$|\partial_0 \psi|^2 = a_C^{-2} \zeta^2 \left| \partial_\tau \dot{\psi} - \beta \partial \dot{\psi} / \partial z_3 \right|^2 + a_C^{-2} |\gamma - \zeta \partial_\tau S + \zeta \beta \partial_z S|^2 \left| \dot{\psi} \right|^2,$$

$$\begin{aligned} |\nabla_x \psi|^2 &= a^{-2} \left| \partial \dot{\psi} / \partial z_1 \right|^2 + a^{-2} \left| \partial \dot{\psi} / \partial z_2 \right|^2 + a^{-2} \left| \partial \dot{\psi} / \partial z_3 \right|^2 + a_C^{-2} (-Z + \beta \gamma)^2 \left| \dot{\psi} \right|^2 \\ &= a^{-2} \left| \nabla_z \dot{\psi} \right|^2 + a_C^{-2} (-Z + \beta \gamma)^2 \left| \dot{\psi} \right|^2. \end{aligned}$$

Note that in $\hat{\Omega}(\hat{\mathbf{r}}(t), R_n)$

$$\left| \partial \dot{\psi} / \partial z_i \right|^2 \leq C (1 + R_n^2) \left| \dot{\psi} \right|^2. \quad (236)$$

Using (228) and (236) we get (234). According to (21)

$$G(|\psi|^2) = -a^{-2} e^{2\sigma} \dot{\psi}^2(z) [-\mathbf{z}^2 + 2\sigma + \ln \pi^{3/2} + 2],$$

yielding the desired inequality (235). ■

Proposition 4.13 *Let ψ be defined by (130), (139) where s, S are defined by (144), (135). Then*

$$\begin{aligned} \mathcal{E}(\hat{\mathbf{r}} + a\mathbf{z}) &= \frac{mc^2}{2} [\zeta^2 (\sigma + \beta z)^2 + 2\zeta^2 \mathbf{z}^2 - \zeta^2 (2\sigma + \ln \pi^{3/2} + 2)] e^{2\sigma} \dot{\psi}^2(z) \\ &\quad + \frac{mc^2}{2} [(\Phi - \gamma)^2 + (\beta \gamma - Z)^2 + 1] e^{2\sigma} \dot{\psi}^2(z) \end{aligned} \quad (237)$$

Proof. Using the change of variables (137), (146) (147), (148) we obtain similarly to (143) and (145)

$$|\tilde{\partial}_t \psi|^2 = \frac{\zeta^2 c^2}{a_C^2} \left| (\partial_\tau - \beta \partial_z) \dot{\psi} \right|^2 + \frac{c^2}{a_C^2} (-\gamma + \Phi)^2 \left| \dot{\psi} \right|^2.$$

The energy density in τ, z variables has the form

$$\begin{aligned} \mathcal{E}(\hat{\mathbf{r}} + a\mathbf{z}) &= \frac{\chi^2}{2m} \left[\frac{\zeta^2}{a_C^2} \left| (\partial_\tau - \beta \partial_z) \dot{\psi} \right|^2 + \frac{\zeta^2}{a_C^2} \left(\left| \nabla_z \dot{\psi} \right|^2 + G \left(\left| \dot{\psi} \right|^2 \right) \right) \right] \\ &+ \frac{\chi^2}{2m} \left[\frac{1}{a_C^2} (-\gamma + \Phi)^2 \left| \dot{\psi} \right|^2 + \frac{1}{a_C^2} (\beta\gamma - Z)^2 \left| \dot{\psi} \right|^2 + \frac{m^2 c^2}{\chi^2} \left| \dot{\psi} \right|^2 \right]. \end{aligned}$$

Using (141) we obtain (237). ■

Proposition 4.14 *Let ζ_n and a_n be related by the formula*

$$\ln \zeta^{-1} = \ln^{1+} a^{-1}. \quad (238)$$

Then conditions (69) and (68) are satisfied.

Proof. To obtain (68) we use Proposition 4.12:

$$\begin{aligned} &a_C^2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\nabla_{0,x} \psi(t, x)|^2 + |G(|\psi(t, x)|^2)| \, d^3x + \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} |\psi(t, x)|^2 \, d^3x \\ &\leq C_2 \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \dot{\psi}^2 \, d^3x = C_2 a^{-3} \int_{\Omega(0, R_n)} \dot{\psi}^2(y/a) \, d^3y \leq C_4 \int_{\mathbb{R}^3} \exp(-|z|^2) \, dz \leq C_5. \end{aligned}$$

To obtain (69) we once again use Proposition 4.12

$$\begin{aligned} &\int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} (a_C^2 |\nabla_{0,x} \psi(t, x)|^2 + a_C^2 |G(|\psi(t, x)|^2)| + |\psi(t, x)|^2) \, d^2\sigma \leq \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} C_2 \dot{\psi}^2 \, d^2\sigma \\ &= \int_{\partial\Omega(\hat{\mathbf{r}}(t), R_n)} C_2 \dot{\psi}^2 \, d^2\sigma = C_3 a^2 a^{-3} \int_{|z|=\bar{\theta}} e^{-|z|^2} \, d^2\sigma = C_4 \exp\left(\ln a^{-1} - \ln^{2/2+} \zeta^{-1}\right) \end{aligned}$$

Condition (238) implies that $\ln a^{-1} = \ln^{1/1+} \zeta^{-1}$ and by (178) $2/2_+ > 1/1_+$. Hence

$$\exp\left(\ln a^{-1} - \ln^{2/2+} \zeta^{-1}\right) \rightarrow 0,$$

and (69) holds. ■

Proposition 4.15 *Let ψ' be defined by (130), (139) where s, S are defined by (144), (135), $R_n = \bar{\theta} a_n \rightarrow 0$. Then the following inequality holds in $\Xi(\bar{\theta})$*

$$\left| \mathcal{E}(\hat{\mathbf{r}} + a\mathbf{z}) - \gamma m c^2 \dot{\psi}^2(\mathbf{z}) \right| \leq C \zeta^{2-\delta} \dot{\psi}^2(\mathbf{z}). \quad (239)$$

Conditions and (90), (71) are satisfied with $\bar{\mathcal{E}}_\infty(t) = \gamma m c^2$. The following estimate holds

$$\left| \rho - q \dot{\psi}^2(\mathbf{z}) \right| \leq C \zeta^{2-\delta} \dot{\psi}^2(\mathbf{z}),$$

and (88) holds as well with $\bar{\rho}_\infty = q$.

Proof. According to (237), since $\gamma^2 + \beta^2\gamma^2 + 1 = 2\gamma^2$, we have

$$\begin{aligned}\mathcal{E}(\hat{\mathbf{r}} + a\mathbf{z}) &= mc^2\gamma^2 e^{2\sigma} \dot{\psi}^2(z) + \frac{mc^2}{2} ((-\gamma + \Phi)^2 - \gamma^2 + (\beta\gamma - Z)^2 - \beta^2\gamma^2) e^{2\sigma} \dot{\psi}^2(\mathbf{z}) \\ &\quad + \frac{mc^2}{2} \zeta^2 [(\sigma + \beta z)^2 + 2\mathbf{z}^2 - (2\sigma + \ln \pi^{3/2} + 2)] e^{2\sigma} \dot{\psi}^2(\mathbf{z}).\end{aligned}$$

Using (158) and (210) we conclude that

$$\begin{aligned}&\frac{mc^2}{2} \left| ((-\gamma + \Phi)^2 - \gamma^2 + (\beta\gamma - Z)^2 - \beta^2\gamma^2) e^{2\sigma} \dot{\psi}^2(\mathbf{z}) \right| \\ &\leq C_0 (|\Phi|(|\Phi| + \gamma) + |Z|(|Z| + |\beta|\gamma)) \dot{\psi}^2(\mathbf{z}) \leq C'_0 |\Phi| \dot{\psi}^2(\mathbf{z}) \leq C''_0 \zeta^{2-\delta} \dot{\psi}^2(\mathbf{z})\end{aligned}\quad (240)$$

One can easily verify that in $\Xi(\bar{\theta})$

$$\begin{aligned}&\frac{mc^2}{2} \zeta^2 \left| [(\sigma + \beta z)^2 + 2\mathbf{z}^2 - (2\sigma + \ln \pi^{3/2} + 2)] e^{2\sigma} \dot{\psi}^2(z) \right| \\ &\leq C_1 \zeta^2 (|\mathbf{z}|^2 + 1) \dot{\psi}^2(z) \leq C'_1 \zeta^{2-\delta} \dot{\psi}^2(\mathbf{z}),\end{aligned}$$

hence (239) holds. Let us estimate now

$$\bar{\mathcal{E}}_n(t) = \int_{\Omega(\hat{\mathbf{r}}(t), R_n)} \mathcal{E} d^3x = a^3 \int_{\Xi(\bar{\theta})} \mathcal{E}(\hat{\mathbf{r}} + a\mathbf{z}) d^3z.$$

Note that

$$a^3 \int_{\Xi(\bar{\theta})} \dot{\psi}^2(\mathbf{z}) d^3z = \pi^{-3/2} \int_{|z| \leq \bar{\theta}} e^{-|z|^2} d^3z = 1 - \pi^{-3/2} \int_{|z| \geq \bar{\theta}} e^{-|z|^2} d^3z. \quad (241)$$

Using (239) and the above formula, we conclude that

$$\bar{\mathcal{E}}_n(t) \rightarrow \gamma mc^2 \quad (242)$$

uniformly for all t and we get (90). Since $\gamma mc^2 > 0$ we get the energy estimate from below (71). To express ρ we use (24) and in (τ, z) -variables obtain

$$\rho = -\frac{\chi q}{mc^2} |\psi|^2 \operatorname{Im} \frac{\tilde{\partial}_t \psi}{\psi} = -\frac{\chi q}{mc^2} e^{2\sigma} \dot{\psi}^2 \frac{c}{a_C} (\Phi - \gamma) = q e^{2\sigma} \dot{\psi}^2 \gamma - q e^{2\sigma} \Phi \dot{\psi}^2.$$

Using (224), (142) and (241) we obtain (88). ■

Theorem 4.16 *Let trajectory $r(t)$ satisfy relations (126)-(129), and let $\varphi(t)$ be defined by (133)-(135) where φ_b is given by (227) and $\varphi_0(t)$ is a given function which satisfies*

$$|\varphi_0| + |\partial_t \varphi_0| + |\partial_t^2 \varphi_0| \leq C. \quad (243)$$

Suppose also ψ to be of the form (130) with ψ_{1D} defined by (139), where the phases $s(t)$, $S(t, y)$ are given by (144), (226). Let $a_n \rightarrow 0$, ζ_n satisfy relations (238), and

$$\theta_n = \bar{\theta} = \ln^{1/2+} \zeta_n^{-1}, \quad \chi_n = mca_{C,n} = mc\zeta_n a_n, \quad R_n = \theta_n a_n. \quad (244)$$

Then ψ is a solution of the KG equation which concentrates at the trajectory $\hat{\mathbf{r}}(t) = (0, 0, r(t))$.

Proof. According to (238) $a^{-1} = \exp \ln^{1/1+} \zeta^{-1}$, hence

$$R_n = \ln^{1/2+} \zeta_n^{-1} \exp \left(-\ln^{1/1+} \zeta_n^{-1} \right) = \exp \left(\ln \ln^{1/2+} \zeta_n^{-1} - \ln^{1/1+} \zeta_n^{-1} \right) \rightarrow 0,$$

implying that the contraction condition (57) holds and we can define concentrating neighborhood $\hat{\Omega}(\hat{\mathbf{r}}, R_n)$ by (56). Note that

$$a_C/a^2 = \zeta/a = \zeta \exp \ln^{1/1+} \zeta^{-1} = \exp \left(\ln^{1/1+} \zeta^{-1} - \ln \zeta^{-1} \right) \leq C,$$

implying that inequality (230) holds. Conditions of Definition 3.2 are satisfied according to Proposition 4.11. Conditions of Definition 3.3 are satisfied as well according to Propositions 4.14 and 4.15. Hence $\psi = \psi_n$ is a solution of the KG equation which concentrates at the trajectory $\hat{\mathbf{r}}(t)$. ■

Remark 4.17 The accelerating force $-q\nabla\varphi$ defined by (135) is of order 1, whereas by (229) the balancing force $-q\varphi_b$ is of order $\zeta^{2-4\delta}$. Hence the balancing force, while preserving the shape $|\psi|$, is vanishingly small compared with the accelerating force.

Remark 4.18 Similarly to solutions of the form (130), (139), (141) we can introduce one more parameter $\gamma_0 = \gamma(t_0)$ and modify (141), (142) as follows:

$$\Psi(t, z) = \pi^{-1/4} e^{\sigma - \gamma_0^2 z^2/2}, \quad \sigma = \ln \gamma^{-1/2} - \ln \gamma_0^{-1/2}. \quad (245)$$

Such a modification introduces a fixed contraction which coincides with the Lorentz contraction at $t_0 = t_0$. All the analysis is quite similar and the corresponding solutions concentrate at $\hat{\mathbf{r}}(t)$. In the case where $r(t) = vt$ for $t < 0$ such a modification with $t_0 = -1$ allows to obtain solutions which coincide with the free uniform solutions with zero electric potential from Section 2.4 for $t < 0$ and can accelerate at positive times.

Remark 4.19 Here we discuss the robustness of the Lorentz contraction in the accelerated motion. Let us consider a rectilinear motion with a trajectory $r(t)$ such that the velocity takes a constant value v_0 for $t < 0$ and a different constant value $v_1 \neq v_0$ for $t \geq T_1 > 0$. Consequently there has to be a non-zero acceleration in the interval $0 < t < T_1$. Let us look at a solution ψ of the form (130), (139), (141) with Ψ of the form (245) and the parameter $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$. Such a solution has a Gaussian shape $|\psi| = \pi^{-3/4} \gamma_0^{1/2} \gamma^{-1/2} e^{-z_1^2/2 - z_2^2/2 - \gamma_0^2 z_3^2/2}$ with the velocity dependent factor $\gamma_0^{1/2} \gamma^{-1/2}$ which does not depend on ζ . When $t < -1$ we have $\varphi = 0$, and ψ is given in $\Omega(\hat{\mathbf{r}}(t), \theta_n a_n)$ by the same formula (45) as the solution for a free charge, with a Gaussian shape $|\psi| = \pi^{-3/4} e^{-z_1^2/2 - z_2^2/2 - \gamma_0^2 z_3^2/2}$. When $t \geq T_1$, the motion is uniform with velocity v_1 , the accelerating part φ_{ac} of the potential is zero and the balancing potential φ_b though not zero when $t \geq T_0$, but it is of order $\zeta^{2-4\delta}$ in $\Omega(\hat{\mathbf{r}}(t), \theta_n a_n)$ and is vanishingly small as $\zeta \rightarrow 0$. For $t \geq T_1$ the Gaussian $|\psi| = \pi^{-3/4} \gamma_0^{1/2} \gamma_1^{-1/2} e^{-z_1^2/2 - z_2^2/2 - \gamma_0^2 z_3^2/2}$ has the same Lorentz contraction factor γ_0 but a different amplitude $\gamma_0^{1/2} \gamma_1^{-1/2}$. The principal part $\omega_0 c^{-2} \gamma v y - \omega_0 t / \gamma$ of the phase \hat{S} in (139) is the same as in (45), and hence it involves Lorentz contraction with factor γ . Consequently, the principal part of the solution for $t \geq T_1$ involves components with different values of the contraction factor. Therefore, the principal part of solution, while translating with the constant velocity v_1 , cannot be obtained by the Lorentz transformation from the solution for a free charge with the velocity v_0 , whereas

the phase can be. Thus, in general, the transition from velocity v_0 to velocity v_1 cannot be reduced to the Lorentz transformation. The fixed Lorentz contraction γ_0 of the Gaussian shape factor is preserved while the velocity changes thanks to the external electric force which causes the acceleration of the charge and also results in the change of the amplitude of the charge distribution.

Observe that Theorem 3.6 can be applied to the considered example, and Einstein's formula is applicable at all times with the same rest mass m . In particular, formula $M = m\gamma$ implies that $M = m\gamma_0$ for $t < 0$ and $M = m\gamma_1$ for $t > T_1$. This observation shows that Einstein's formula holds all the time and fully applies to accelerating regimes, whereas the Lorentz contraction formula can be applied only to some characteristics of an accelerating charge distribution. Such a difference is specific to accelerating regimes and is in a sharp contrast with the case of a global uniform motion without external forces.

5 Concentration of solutions of a linear KG equation

The results of Section 3 on concentrating solutions are directly applicable to solutions of the linear KG equation by setting $G = 0$. In this case Theorem 3.6 can be applied, and solutions of the linear KG equations can only concentrate at trajectories which satisfy relativistic point equations. The size parameter a now is not involved in the equation but rather describes the localization of a sequence of solutions of the linear equations. The significant difference is that the linear KG equations do not have global localized solutions as described in Sections 2.3 and 2.4. Consequently, one cannot simply apply the relativistic Einstein argument for the 4-vector of the global energy-momentum to the linear KG equation. As to the results of Section 4 they can be modified for the linear case as follows. For a given trajectory $r(t)$ wave function ψ is defined by the same formulas (139), (130). The identity (155) cannot be used, and in Q defined by (154) the term $\zeta^2 \partial_z^2 \Psi$ will not cancel with G' , resulting in replacing the term $\beta^2 \zeta^2 (z^2 - 1)$ in (156) by the term $\gamma^{-2} \zeta^2 (z^2 - 1)$; the term $2\zeta^2 \sigma$ in (156) is absent. These arguments show that Q is a small perturbation of order ζ^2 . Since we do not use the structural details of Q but only its smallness, all the estimates in Section 4 can be carried out with this modification. Hence, a small (of order $\zeta^{2-4\delta}$) balancing potential φ_b exists in the linear case as well. Analyzing estimates made in Proposition 4.15, one finds that the nonlinearity produces a vanishing contribution to E_∞ . Thus the following modification of Theorem 4.16 holds.

Theorem 5.1 *Let trajectory $r(t)$ satisfy (126)-(129), and let the KG equation be linear, namely $G' = 0$ in (3) and (22), (26). Suppose ψ to be of the form (130) with ψ_{1D} defined by (139), where the phases $s(t)$, $S(t, y)$ are given by (144) (226). Let $a_n \rightarrow 0$, ζ_n satisfy (238), χ_n , R_n , θ_n be defined by (244), and $\varphi(t)$ be defined by (133)-(135) where φ_b is given by (227) and $\varphi_0(t)$ is a given function satisfying (243). Then ψ is a solution of the linear KG equation which concentrates at the trajectory $\hat{\mathbf{r}}(t) = (0, 0, r(t))$.*

6 Appendix

For completeness, we present here verification of the energy and momentum conservation equations. To verify the energy conservation equation, we multiply (3) by $\tilde{\partial}_t^* \psi^*$ where $\tilde{\partial}_t$ is

defined by (4). We obtain

$$\begin{aligned} & -c^{-2}\partial_t \left(\tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* \right) + \nabla \left(\nabla \psi \tilde{\partial}_t^* \psi^* + \nabla \psi^* \tilde{\partial}_t \psi \right) \\ & - \left(\nabla \psi \nabla \tilde{\partial}_t^* \psi^* + \nabla \psi^* \nabla \tilde{\partial}_t \psi \right) - \partial_t G(\psi^* \psi) - \kappa_0^2 \partial_t \psi \psi^* = 0 \end{aligned}$$

and rewrite in the form

$$\begin{aligned} \partial_t \left(c^{-2} \left(\tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* \right) + \nabla \psi \nabla \psi^* + G(\psi^* \psi) + \kappa_0^2 \psi \psi^* \right) = \\ \nabla \left(\nabla \psi \tilde{\partial}_t^* \psi^* + \nabla \psi^* \tilde{\partial}_t \psi \right) - 2m\chi^{-2} \nabla \varphi \cdot \mathbf{J}. \end{aligned} \quad (246)$$

Finally, using the definition of $\tilde{\partial}_t$, \mathbf{J} and \mathbf{P} , we rewrite it in the form of the energy conservation equation (32).

To obtain the momentum conservation equation (33), we multiply equation (3) by $\nabla \psi^*$, and then using the identity $v\tilde{\partial}_t u + u\tilde{\partial}_t^* v = \partial_t(uv)$, we obtain

$$-\frac{1}{c^2} \partial_t \left(\tilde{\partial}_t \psi \nabla \psi^* \right) + \frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \nabla \psi^* + \tilde{\nabla}^2 \psi \tilde{\nabla}^* \psi^* - G'(\psi^* \psi) \psi \tilde{\nabla}^* \psi^* - \kappa_0^2 \psi \tilde{\nabla}^* \psi^* = 0.$$

Obviously, $\tilde{\partial}_t(\nabla \psi) = \nabla \tilde{\partial}_t \psi - \frac{iq}{\chi} \nabla \varphi \psi$. Therefore, we arrive at the equation

$$\begin{aligned} \partial_t \mathbf{P} + \frac{\chi^2}{2m} \frac{1}{c^2} \left(\tilde{\partial}_t \psi \left(\nabla \tilde{\partial}_t^* \psi + \frac{iq}{\chi} \nabla \varphi \psi^* \right) + \tilde{\partial}_t^* \psi^* \left(\nabla \tilde{\partial}_t \psi - \frac{iq}{\chi} \nabla \varphi \psi \right) \right) \\ + \frac{\chi^2}{2m} \left(\nabla^2 \psi \nabla \psi^* + \nabla^2 \psi^* \nabla \psi \right) - \frac{\chi^2}{2m} \nabla \left(G(\psi^* \psi) + \kappa_0^2 \psi^* \psi \right) = 0. \end{aligned} \quad (247)$$

Substituting the identity

$$\left(\nabla^2 \psi \nabla \psi^* + \nabla^2 \psi^* \nabla \psi \right) = \partial_j \cdot \left(\partial_j \psi \nabla \psi^* + \partial_j \psi^* \nabla \psi \right) - \nabla \left(\nabla \psi \cdot \nabla \psi^* \right) \quad (248)$$

into (247) we obtain the *momentum conservation law* for l -th component of \mathbf{P} in the following form

$$\begin{aligned} \partial_t \mathbf{P}_l + \frac{\chi^2}{2m} \frac{1}{c^2} \nabla_l \left(\tilde{\partial}_t \psi \tilde{\partial}_t^* \psi \right) + \rho \nabla \varphi - \frac{\chi^2}{2m} \nabla_l \left(G(\psi^* \psi) + \kappa_0^2 \psi^* \psi \right) \\ + \frac{\chi^2}{2m} \left(\sum_j \nabla_j \left(\tilde{\nabla}_j \psi \tilde{\nabla}_l^* \psi^* + \tilde{\nabla}_j^* \psi^* \tilde{\nabla}_l \psi \right) - \nabla_l \sum_j \left(\tilde{\nabla}_j \psi \tilde{\nabla}_j^* \psi^* \right) \right) = 0. \end{aligned}$$

Finally, using (22), we rewrite the the above equation in the form (33).

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